Dovilė MERKEVIČIŪTĖ

OPTIMIZATION OF ELASTIC-PLASTIC SYSTEMS UNDER STIFFNESS AND STABILITY CONSTRAINTS AT SHAKEDOWN

Summary of Doctoral Dissertation
Technological Sciences, Civil Engineering (02 T)

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Scientific Supervisor

Prof Dr Habil **Juozas ATKOČIŪNAS** (Vilnius Gediminas Technical University, Technological Sciences, Civil Engineering – 02 T)

The doctoral dissertation is defended at the Council of Scientific Field of Civil Engineering at Vilnius Gediminas Technical University:

Chairman

Prof Dr Habil **Gintaris KAKLAUSKAS** (Vilnius Gediminas Technical University, Technological Sciences, Civil Engineering – 02 T)

Members:

Prof Dr Habil **Vytautas STANKEVIČIUS** (Kaunas University of Technology, Technological Sciences, Civil Engineering – 02 T)

Prof Dr Habil **Romualdas MAČIULAITIS** (Vilnius Gediminas Technical University, Technological Sciences, Civil Engineering – 02 T)

Prof Dr Habil **Vytautas Jonas STAUSKIS** (Vilnius Gediminas Technical University, Technological Sciences, Civil Engineering – 02 T)

Prof Dr Habil **Antanas ŽILIUKAS** (Kaunas University of Technology, Technological Sciences, Mechanical Engineering – 09 T)

Opponents:

Prof Dr Habil **Jonas BAREIŠIS** (Kaunas University of Technology, Technological Sciences, Mechanical Engineering – 09 T)

Prof Dr Habil **Rimantas KAČIANAUSKAS** (Vilnius Gediminas Technical University, Technological Sciences, Civil Engineering – 02 T)

The dissertation will be defended at the public meeting of the Council of Scientific Field of Civil Engineering in Senate Hall of Vilnius Gediminas Technical University at 2 p. m. on December 16, 2005.
Address: Saulėtekio al. 11, LT-10223 Vilnius-40, Lithuania
Tel. +370 5 274 49 52, +370 5 274 49 56; fax +370 5 270 01 12,
e-mail doktor@adm.vtu.lt

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VILNIAUS GEDIMINO TECHNIKOS UNIVERSITETAS

Dovilė MERKEVIČIŪTĖ

TAMPRIŲ-PLASTINIŲ PRISITAIKANČIŲ SISTEMŲ OPTIMIZACIJA SU STANDUMO IR STABILUMO SĄLYGOMIS

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Mokslinis vadovas

prof. habil. dr. **Juozas ATKOČIŪNAS** (Vilniaus Gedimino technikos universitetas, technologijos mokslai, statybos inžinerija – 02 T).

Disertacija ginama Vilniaus Gedimino technikos universiteto Statybos inžinerijos mokslo krypties taryboje:

Pirmininkas

prof. habil. dr. **Gintaris KAKLAUSKAS** (Vilniaus Gedimino technikos universitetas, technologijos mokslai, statybos inžinerija – 02 T).

Nariai:

prof. habil. dr. **Vytautas STANKEVIČIUS** (Kauno technologijos universitetas, technologijos mokslai, statybos inžinerija – 02 T),
prof. habil. dr. **Romualdas MAČIULAITIS** (Vilniaus Gedimino technikos universitetas, technologijos mokslai, statybos inžinerija – 02 T),
prof. habil. dr. **Vytautas Jonas STAUSKIS** (Vilniaus Gedimino technikos universitetas, technologijos mokslai, statybos inžinerija – 02 T),
prof. habil. dr. **Antanas ŽILIUKAS** (Kauno technologijos universitetas, technologijos mokslai, mechanikos inžinerija – 09 T).

Oponentai:

prof. habil. dr. **Jonas BAREIŠIS** (Kauno technologijos universitetas, technologijos mokslai, mechanikos inžinerija – 09 T),
prof. habil. dr. **Rimantas KAČIANAUSKAS** (Vilniaus Gedimino technikos universitetas, technologijos mokslai, statybos inžinerija – 02 T).

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Adresas: Saulėtekio al. 11, LT-10223 Vilnius - 40, Lietuva.
Tel. +370 5 274 49 52, +370 5 274 49 56; faksas +370 5 270 01 12,
el. paštas. doktor@adm.vtu.lt

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General Characteristics of the Dissertation

Field and need of research. Optimization problems (to which is dedicated this dissertation) of structural mechanics are introductory stage of structure optimum design based on principles of solid deformable body mechanics, mathematical programming theory, its methods and their mechanical interpretation. In order to base calculation on real operating conditions of structure, it is necessary evaluate as exact as possible structure material properties, external effects and other factors in mathematical models of optimization problems. Partially it is achieved by including plastic properties of material. Calculation and design of the structures, taking in to account plastic strains, allows to use their bearing capacity more efficiently and make more economic project (in this dissertation research is developed on the basis of perfect plasticity theory). From the other side, real effect for structure are often cyclic (variable repeated load character is also evaluated in this work). In the dissertation it is assumed that load is quasi–static and is characterised by load variation bounds (deterministic formulation of problems is considered).

Under repeated loading a structure can lose its serviceability because of its progressive plastic failure or because of alternating strain (usually both cases are called cyclic–plastic collapse). But, if residual forces together with variable part that do not violate the admissible bounds appear in the initial stage of loading, the structure adapts to existing load and further behaves elastically. This phenomenon is considered in shakedown theory. Applying the classical Melan and Koiter theorems, it is possible to analyse only simple systems at shakedown by elementary methods. Meanwhile for civil engineering, calculation of any complexity elastic–plastic structures subjected by variable repeated load is relevant. That had influence on the choice of research aspect of the dissertation: *optimization of elastic–plastic systems, subjected by variable repeated load, under stiffness and stability constraints, applying extremum energy principles, theory of mathematical programming, numerical methods and computer technologies.* Growing number of scientific works dedicated to adapted structure calculation shows importance of these researches. But there is especially small number of works concerning optimisation of adapted structures under stiffness and stability constraints. Solution of structure optimization problems at shakedown is difficult as stress–strain state of dissipative systems depends on loading history. Not only general point of adapted structure analysis and optimization theory is solved in dissertation, but also new methods and algorithms of bar–structure optimization are presented. That has also practical value in civil engineering.

Main objective. Further development of theory of elastic–plastic adapted system optimization, creation of new calculation methods and algorithms.

Main tasks: 1) to review calculation methods of systems at shakedown; 2) to formulate extremum energy principles for perfectly elastic–plastic discrete systems at shakedown; 3) to construct general non–linear mathematical models of analysis and optimization problems; 4) to bring out connection between extremum problems of adapted system optimization and theory of non–linear mathematical programming; 5) to perform optimization of bar–structures taking in to account stiffness and stability conditions according to requirements for the second group of limit states: mathematical models of optimum design and load oriented problems, solution methods and numerical experiments; 6) to create algorithm to solve structure optimization problems, which do not have full initial information; 7) to create technique of incremental analysis for unloading phenomenon fixation.

Scientific novelty. 1) New potential, which is provided by connections between mathematical programming and extremum energy principles, are shown for formulation of analysis and optimization problems of shakedown theory and their numerical solution. 2) New non–linear mathematical models of load oriented and optimum design problems with constraints ensuring shakedown and serviceability of structure are created. 3) It is determined that Haar–Kármán principle is suitable only for systems with holonomic behaviour. 4) It is proved that residual displacements are varying non–monotonically. Dual problems of mathematical programming can not be applied for analysis of stress–strain state of unloading system. 5) It is proved that complementary slackness condition simulates only holonomic deformation process during shakedown of structure. 6) It is shown application possibilities of strain compatibility equations for formulation of new mathematical models of residual displacement variation bound calculation problems. 7) Technique of bar–structure optimization is implemented: new non–linear mathematical models for frames (under strength and stiffness constraints) and trusses (with stability conditions (EC3)) optimization problems, solution algorithms, numerical experiments are carried out. 8) Optimization technique is created for trusses subjected by moving load. 9) Technique of incremental analysis for unloading phenomenon fixation is created; numerical experiment is carried out for bending plate with non–linear von Mises yield conditions.

Approbation and publications. The main results of this work were submitted in 10 scientific conferences. Thirteen papers were published on the topic of dissertation: 6 of them were published in the acknowledged editions, 5 – in proceedings of international conferences, 2 – in proceedings of republican conferences.

The scope of the thesis. The Lithuanian written thesis consists of introduction, five main chapters, conclusions and a list of references. The total scope of the dissertation – 131 pages, 36 pictures and 10 tables.

Content of the Work

1. Review of Calculation Methods of the Structures at Shakedown

This chapter reviews methods of adapted system calculation. The important part is comparative analysis of displacement calculation methods after shakedown.

2. General Mathematical Models of Optimization Problems

The main definitions and dependencies of discretized systems. Discrete model of the structure, which degree of freedom is m, consists of s ($k=1,2,...,s$, $k\in K$) finite elements. Each k element has s_k nodal points ($l=1,2,...,s_k$, $l\in L$). The total number of design sections is $\zeta \le s\times s_k$. Nodal internal forces of element compound one n–vector of discrete model forces $\boldsymbol{S}=\left(\boldsymbol{S}_1,\boldsymbol{S}_2,...,\boldsymbol{S}_v,..., \boldsymbol{S}_\zeta\right)^{\mathrm{T}}=\left(S_z\right)^{\mathrm{T}}$ and strains – n–vector $\boldsymbol{\Theta}=\left(\boldsymbol{\Theta}_1,\boldsymbol{\Theta}_2,...,\boldsymbol{\Theta}_v, ...,\boldsymbol{\Theta}_\zeta\right)^{\mathrm{T}}=\left(\boldsymbol{\Theta}_z\right)^{\mathrm{T}}$, $v=1,2,...,\zeta$ ($v\in V$), $z=1,2,...,n$. Load $\boldsymbol{F}(t)$ is characterized by time–, t–, independent variation bounds $\boldsymbol{F}_{sup}=\left(F_{1,sup},F_{2,sup}, ...,F_{m,sup}\right)^{\mathrm{T}}$, $\boldsymbol{F}_{inf}=\left(F_{1,inf},F_{2,inf},...,F_{m,inf}\right)^{\mathrm{T}}$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$). Equilibrium $\boldsymbol{A}\boldsymbol{S}=\boldsymbol{F}$ and geometrical $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}-\boldsymbol{D}\boldsymbol{S}=\boldsymbol{0}$ equations do not depend on characteristics of system material and can be written via residual quantities as follow:

$$\boldsymbol{A}\boldsymbol{S}_r=\boldsymbol{0}, \quad \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_r=\boldsymbol{\Theta}_r, \quad \boldsymbol{\Theta}_r=\boldsymbol{D}\boldsymbol{S}_r+\boldsymbol{\Theta}_p. \tag{2.1}$$

Elastic displacements $\boldsymbol{u}_e(t)$ and forces $\boldsymbol{S}_e(t)$ of the structure are determined using influence matrixes of displacements and forces, $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$, respectively:

$$\boldsymbol{u}_e(t)=\boldsymbol{\beta}\boldsymbol{F}(t),\ \ \boldsymbol{S}_e(t)=\boldsymbol{\alpha}\boldsymbol{F}(t),\ \ \boldsymbol{\beta}=\left(\boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^{\mathrm{T}}\right)^{-1},\ \ \boldsymbol{\alpha}=\boldsymbol{K}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{\beta},\ \ \boldsymbol{K}=\boldsymbol{D}^{-1}. \tag{2.2}$$

The number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ is $p=2^m$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}_j \le \boldsymbol{F}_{sup}$):

$$\boldsymbol{S}_{ej}=\boldsymbol{\alpha}\boldsymbol{F}_j,\ j=1,2,...,p,\ (j\in J). \tag{2.3}$$

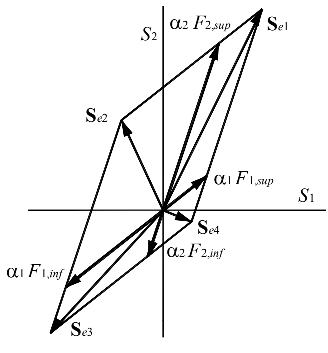

Fig 1. Locus of elastic forces

In the case of two loads F_1, F_2, domain of elastic force variation (locus) is shown in Fig 1. Plasticity constant C of elastic–plastic structure relates to dimensions and material yield limit σ_y (limit force S_0) of discrete model. Limit force S_{0k} ($k\in K$) is assumed as constant in whole finite element. Non–linear yield conditions

$$\varphi_{kl,j}=C_k-f_{kl,j}\left(\boldsymbol{S}_{ekl,j}+\boldsymbol{S}_{rkl}\right)\ge 0, \quad k\in K,\ \ l\in L,\ \ j\in J \tag{2.4}$$

are verified in each nodal point l of the k–th finite element for all load combinations $j \in J$. For the entire structure the non–linear yield conditions $\boldsymbol{f}(\mathrm{S}) \le \boldsymbol{C}$ can be written as follows:

$$\boldsymbol{f}_j\left(\boldsymbol{S}_{ej} + \boldsymbol{S}_r\right) \le \boldsymbol{C}, \quad j \in J. \tag{2.5}$$

In that case, direct analysis of loading history is avoided.

Statically admissible residual forces $\boldsymbol{S}_r$ satisfy equilibrium equations (2.1) and yield conditions (2.4). Kinematically possible residual displacements $\boldsymbol{u}_r$ satisfy geometrical equations (2.1): $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_r = \boldsymbol{D}\,\boldsymbol{S}_r + \boldsymbol{\Theta}_p$. Components of vector $\boldsymbol{\Theta}_p = \left(\boldsymbol{\Theta}_{pkl}\right)^{\mathrm{T}}$ are calculated according to formula:

$$\boldsymbol{\Theta}_{pkl} = \sum_j \left[\nabla\varphi_{kl,j}\,(\boldsymbol{S}_{ekl,j} + \boldsymbol{S}_{rkl})\right]^{\mathrm{T}} \lambda_{kl,j}\,,\ \lambda_{kl,j} \ge 0\,;\ k \in K\,,\ l \in L\,,\ j \in J\,. \tag{2.6}$$

Residual strains $\boldsymbol{\Theta}_r = \boldsymbol{D}\,\boldsymbol{S}_r + \boldsymbol{\Theta}_p$ and displacements $\boldsymbol{u}_r$ of structure at shakedown can be non–unique: they depend on the particular loading history $\boldsymbol{F}(t)$. If load is defined only by variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$, the calculation of exact values of residual displacements becomes problematical because of unloading phenomenon appearing at cross–sections: then displacements $\boldsymbol{u}_r$ are varying non–monotonically, it is possible to determine only their lower $\boldsymbol{u}_{r,inf}$ and upper $\boldsymbol{u}_{r,sup}$ variation bounds ($\boldsymbol{u}_{r,inf} \le \boldsymbol{u}_r(t) \le \boldsymbol{u}_{r,sup}$).

Extremum energy principles and analysis problem

Static formulation of analysis problem (principle of minimum complementary energy) reads: find

$$\min\ \widetilde{\mathcal{F}}(\boldsymbol{x}) = \frac{1}{2}\sum_k \boldsymbol{S}_{rk}^{\mathrm{T}}\boldsymbol{D}_k\,\boldsymbol{S}_{rk} = a^*, \tag{2.7}$$

subject to

$$\sum_k \boldsymbol{A}_k\boldsymbol{S}_{rk} = \boldsymbol{0}\,, \quad \boldsymbol{S}_{rk} = \left(\boldsymbol{S}_{rk1}, \boldsymbol{S}_{rk2}, \ldots, \boldsymbol{S}_{rkl}, \ldots, \boldsymbol{S}_{rks_k}\right)^{\mathrm{T}}, \tag{2.8}$$

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\boldsymbol{S}_{ekl,j} + \boldsymbol{S}_{rkl}\right) \ge 0\,,\ C_k = \left(S_{0k}\right)^2,\ k \in K\,,\ l \in L\,,\ j \in J\,. \tag{2.9}$$

Vectors of limit forces $\boldsymbol{S}_0 = \left(S_{01}, S_{02}, \ldots, S_{0k}, \ldots, S_{0\eta}\right)^{\mathrm{T}}$ and quasi–elastic forces $\boldsymbol{S}_{ej}$ are known. As functions $\varphi_{kl,j} \ge 0$ are convex and matrix $\boldsymbol{D}_k$ is positively defined, the optimal solution $\boldsymbol{S}_r^*$ of non–linear analysis problem (2.7) – (2.9) is global.

Matrix $\boldsymbol{B} = \left[-\boldsymbol{A}''^{\mathrm{T}} \left(\boldsymbol{A}'^{\mathrm{T}} \right)^{-1}, \ \boldsymbol{I} \right]$ allows analysis problem (2.7) – (2.9) rewrite as follows: find

$$\min \ \frac{1}{2} \sum_{k} \boldsymbol{S}_r''^{\mathrm{T}} \boldsymbol{B}_k \, \boldsymbol{D}_k \, \boldsymbol{B}_k^{\mathrm{T}} \boldsymbol{S}_r'' = \min \ \frac{1}{2} \sum_{k} \boldsymbol{S}_r''^{\mathrm{T}} \widetilde{\boldsymbol{D}}_k \boldsymbol{S}_r'' , \qquad (2.10)$$

subject to

$$\varphi_{kl,j} = C_k - f_{kl,j} \left(\boldsymbol{S}_{ekl,j} + \boldsymbol{B}_{kl}^{\mathrm{T}} \boldsymbol{S}_r'' \right) \geq 0 , \ C_k = \left(S_{0k} \right)^2 , \ k \in K , \ l \in L , \ j \in J . \qquad (2.11)$$

Optimal solutions $\boldsymbol{S}_r''^{*}$ of the problem (2.10) – (2.11) are unknowns of the force method.

Kinematic formulation of analysis problem (principle of minimum total potential energy) reads: find

$$\max \left\{ -\frac{1}{2} \sum_{k} \boldsymbol{S}_{rk}^{\mathrm{T}} \boldsymbol{D}_k \boldsymbol{S}_{rk} - \sum_{k} \sum_{j} \boldsymbol{\lambda}_{k,j}^{\mathrm{T}} \nabla \boldsymbol{\varphi}_{k,j} \left(\boldsymbol{S}_{k,j} \right) \boldsymbol{S}_{rk} - \sum_{k} \sum_{l} \sum_{j} \lambda_{kl,j} \left(C_k - f_{kl,j} \left(\boldsymbol{S}_{k,j} \right) \right) \right\} , \qquad (2.12)$$

subject to

$$\boldsymbol{D}_k \boldsymbol{S}_{rk} + \sum_{j} \left[\nabla \varphi_{k,j} \left(\boldsymbol{S}_{k,j} \right) \right]^{\mathrm{T}} \boldsymbol{\lambda}_{k,j} - \boldsymbol{A}_k^{\mathrm{T}} \boldsymbol{u}_r = \boldsymbol{0} , \ \ \boldsymbol{\lambda}_{k,j} \geq \boldsymbol{0} , \qquad (2.13)$$

$$\boldsymbol{S}_{k,j} = \left(\boldsymbol{S}_{ek,j} + \boldsymbol{S}_{rk} \right), \quad k \in K , \quad l \in L , \quad j \in J . \qquad (2.14)$$

Unknowns of the problem (2.12) – (2.14), which is dual to (2.7) – (2.9), are residual forces $\boldsymbol{S}_r$, displacements $\boldsymbol{u}_r$ and plasticity multipliers $\boldsymbol{\lambda}_j$, $j \in J$.

Complementary slackness conditions of the mathematical programming

$$\lambda_{kl,j} \left(C_k - f_{kl,j} \left(\boldsymbol{S}_{k,j} \right) \right) = 0 , \ \ \lambda_{kl,j} \geq 0 , \ \ k \in K , \ \ l \in L , \ \ j \in J , \qquad (2.15)$$

which are included in to objective function of the problem (2.12) – (2.14), do not allow direct evaluation of unloading phenomenon (it appears when exists $\lambda_i > 0$ and $\varphi_i > 0$, $i = 1, 2, ..., \zeta$, $i \in I$ during deformation process). Optimal solution $\boldsymbol{S}_r^*$, $\boldsymbol{u}_r^*$, $\boldsymbol{\lambda}_j^*$ ($j \in J$) of the problem (2.12) – (2.14) is obtained without consideration of loading history. Nevertheless, particular loading history exists $\boldsymbol{F}(t)$ ($\boldsymbol{F}_{inf} \leq \boldsymbol{F}(t) \leq \boldsymbol{F}_{sup}$), which leads structure to shakedown with $\boldsymbol{S}_r^*$, $\boldsymbol{u}_r^*$ and $\boldsymbol{\lambda}_j^*$. It becomes obvious, that the mathematical model of analysis problem (2.7) – (2.9) of structure at shakedown can be obtained according to Haar–Kármán principle (structure with holonomic behaviour).

Residual strain compatibility equations are obtained after elimination of residual displacements $\boldsymbol{u}_r$ from geometrical equations (2.13):

$$-\boldsymbol{B\Theta}_p = \boldsymbol{B}_r \boldsymbol{S}_r \ , \qquad (2.16)$$

$$\boldsymbol{\Theta}_{pkl} = \sum_j \left[\nabla \varphi_{kl,j} \left(\boldsymbol{S}_{ekl,j} + \boldsymbol{S}_{rkl} \right) \right]^{\mathrm{T}} \lambda_{kl,j} \ , \ \lambda_{kl,j} \geq 0 \ ; \quad k \in K \ , \ l \in L \ , \ j \in J \ . \quad (2.17)$$

Here $\boldsymbol{B}_r = -\boldsymbol{A}''^{\mathrm{T}} \left(\boldsymbol{A}'^{\mathrm{T}} \right)^{-1} \boldsymbol{D}' + \boldsymbol{D}''$. Unknowns of the problem (2.12) – (2.16), (2.17) become residual forces $\boldsymbol{S}_r$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_j$, $j \in J$.

Problem of optimal design. This work is more oriented to frames and beams with I–shape section or plates with three–layered, “sandwich” type cross–section. Residual displacements $\boldsymbol{u}_r$ and forces $\boldsymbol{S}_r$ are related to vector of plasticity multipliers $\boldsymbol{\lambda}$ ($\boldsymbol{\Theta}_p = \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\lambda}$) by influence matrixes $\boldsymbol{H}$ and $\boldsymbol{G}$:

$$\boldsymbol{u}_r = \overline{\boldsymbol{H}} \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\lambda} = \boldsymbol{H} \boldsymbol{\lambda} \ , \quad \boldsymbol{S}_r = \overline{\boldsymbol{G}} \boldsymbol{\Phi}^{\mathrm{T}} \boldsymbol{\lambda} = \boldsymbol{G} \boldsymbol{\lambda} \ , \qquad (2.18)$$

$$\overline{\boldsymbol{H}} = \left(\boldsymbol{AKA}^{\mathrm{T}} \right)^{-1} \boldsymbol{AK} \ , \qquad \overline{\boldsymbol{G}} = \boldsymbol{KA}^{\mathrm{T}} \overline{\boldsymbol{H}} - \boldsymbol{K} \ .$$

Here $\boldsymbol{\Phi}$ – matrix of peace–wise linearized yield conditions $\boldsymbol{\Phi} \left(\boldsymbol{S}_r + \boldsymbol{S}_{ej} \right) \leq \boldsymbol{S}_0$.

Structure optimal design problem is stated as follows:
when load variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$ are prescribed, the vector of structure limit forces $\boldsymbol{S}_0$, satisfying optimality criterion $\min \psi \left(\boldsymbol{S}_0 \right)$, *strength and stiffness conditions, are to be found.*

Mathematical models of structure optimal design problem are presented in Table 1: (2.19) – (2.22) has linear, (2.23) – (2.26) has nonlinear yield conditions.

In case of ideal cross–sectional form and homogenous yield condition, mini-

Table 1. Mathematical models of structure optimal design problem

Linear yield conditions	Non–linear yield conditions
find $\min \ \psi \left(\boldsymbol{S}_0 \right) = \min \ \boldsymbol{L}^{\mathrm{T}} \boldsymbol{S}_0$, (2.19) subject to $\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi} \left(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej} \right) \geq \boldsymbol{0}$, (2.20) $\boldsymbol{\lambda}_j^{\mathrm{T}} \boldsymbol{\varphi}_j = 0$, $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, (2.21) $\boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j$, $j \in J$, $\boldsymbol{u}_{r,\min} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,\max}$.(2.22)	find $\min \ \boldsymbol{L}^{\mathrm{T}} \boldsymbol{S}_0$, (2.23) subject to $\min \ \widetilde{\mathscr{F}} \left(\mathrm{S}_r \right) = \min \ \frac{1}{2} \boldsymbol{S}_r^{\mathrm{T}} \boldsymbol{D} \boldsymbol{S}_r$, (2.24) $\boldsymbol{A} \boldsymbol{S}_r = \boldsymbol{0}$, $\boldsymbol{\varphi}_j = \boldsymbol{C} - \boldsymbol{f}_j \left(\boldsymbol{S}_r + \boldsymbol{S}_{ej} \right) \geq \boldsymbol{0}$, $j \in J$, $\boldsymbol{C} = \boldsymbol{C} \left(\boldsymbol{S}_0 \right)$, $\boldsymbol{S}_0 \geq \boldsymbol{0}$, (2.25) $\boldsymbol{u}_{r,\min} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,\max}$. (2.26)

mum weight correlates with linear objective function $\min \boldsymbol{L}^{\mathrm{T}}\boldsymbol{S}_0$. Here $\boldsymbol{L}$ is a vector of weight factors of optimality criterion (2.19) and (2.23). In the case of minimum volume structure design, all properties of material are prescribed, only dimensions of element or cross–section can vary. Thus, vector of elastic forces $\boldsymbol{S}_e$ is varying and has to be recalculated in case of new geometry of cross–section. In the problems (2.19) – (2.22) and (2.23) – (2.26) unknowns are vectors of limit forces $\boldsymbol{S}_0$ and plastic multipliers $\boldsymbol{\lambda}$. Vector $\boldsymbol{\lambda}$ can be determined applying Rosen optimality criterion (in dissertation project gradient method is used for non–linear problem solution):

$$\boldsymbol{\lambda} = \left(\nabla\boldsymbol{\varphi}\left(\boldsymbol{S}^*\right)\left[\nabla\boldsymbol{\varphi}\left(\boldsymbol{S}^*\right)\right]^{\mathrm{T}} \right)^{-1} \nabla\boldsymbol{\varphi}\left(\boldsymbol{S}^*\right)\nabla\widetilde{\mathscr{F}}\left(\mathrm{S}_r^*\right). \tag{2.27}$$

Here total forces $\boldsymbol{S}^* = \boldsymbol{S}_r^* + \boldsymbol{S}_e^*$ are calculated using optimal solution $\boldsymbol{S}_r^*$ of analysis problem (2.7) – (2.9): $\nabla\widetilde{\mathscr{F}}\left(\mathrm{S}_r^*\right)$ is gradient of objective function (2.7). When yield condition is not linear, matrix $\boldsymbol{G}$ depends on matrix of gradients $\nabla = \left[\varphi\left(\boldsymbol{S}_r + \boldsymbol{S}_e\right)\right]^{\mathrm{T}}$, which is related with $\boldsymbol{S}_r$.

In the constraints (2.22), (2.26) vectors $\boldsymbol{u}_{r,\max}$, $\boldsymbol{u}_{r,\min}$ are known admissible residual displacements. In constraints (2.22), (2.26) it is not difficult to evaluate results of elastic calculation using vectors $\boldsymbol{u}_{e,sup} = \boldsymbol{\beta}_{sup}\boldsymbol{F}_{sup} + \boldsymbol{\beta}_{inf}\boldsymbol{F}_{inf}$, $\boldsymbol{u}_{e,inf} = \boldsymbol{\beta}_{sup}\boldsymbol{F}_{inf} + \boldsymbol{\beta}_{inf}\boldsymbol{F}_{sup}$:

$$\boldsymbol{u}_{\min} \le \boldsymbol{u}_{e,inf} + \boldsymbol{u}_{r,inf}, \quad \boldsymbol{u}_{e,sup} + \boldsymbol{u}_{r,sup} \le \boldsymbol{u}_{\max}. \tag{2.28}$$

Mathematical model of load optimization problem. General formulation of load optimization problem is stated as follows:

when limit force vector $\boldsymbol{S}_0$ *is prescribed, load variation bounds* $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$, *satisfying optimality criterion* $\max\left(\boldsymbol{T}_{sup}^{\mathrm{T}}\boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^{\mathrm{T}}\boldsymbol{F}_{inf}\right)$, *shakedown and stiffness conditions, are to be found.*

Here $\boldsymbol{T}_{sup}$, $\boldsymbol{T}_{inf}$ are vectors of weight factors of optimality criterion. In case of non–linear yield conditions following mathematical model of load optimization problem corresponds aforementioned formulation: find

$$\max\left(\boldsymbol{T}_{sup}^{\mathrm{T}}\boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^{\mathrm{T}}\boldsymbol{F}_{inf}\right), \tag{2.29}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{C} - \boldsymbol{f}_j\left(\boldsymbol{S}_r + \boldsymbol{S}_{ej}\right) \ge \mathbf{0}, \tag{2.30}$$

$$\boldsymbol{\lambda}_j^{\mathrm{T}}\,\boldsymbol{\varphi}_j = 0\,,\;\; \boldsymbol{\lambda}_j \geq \mathbf{0}\,,\;\; \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j\,,\;\; j \in J\,, \tag{2.31}$$

$$\boldsymbol{F}_{sup} \geq \mathbf{0}\,, \quad -\boldsymbol{F}_{inf} \geq \mathbf{0}\,, \tag{2.32}$$

$$\boldsymbol{u}_{r,\min} \leq \boldsymbol{u}_{r,inf}\,, \qquad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,\max}\,. \tag{2.33}$$

Vectors of load variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$ and plastic multipliers $\boldsymbol{\lambda}$ are unknowns of the problem (2.29) – (2.33). Constraints (2.30) ensure shakedown of structure. Displacement constraints (2.33) are similar to those in the problem (2.19) – (2.23).

3. Optimal design of frames at shakedown

Frame analysis problem. The characteristics of frame cross–sectional resistance are limit bending moment $M_0 = \sigma_y W_{pl}$ and axial force $N_0 = \sigma_y A$; here W_{pl} is plastic modulus of section, A is cross–sectional area. For cross–section with ideal shape, linear yield conditions read (Fig 2):

$$|M| + c|N| \leq M_0\,, \qquad c = \frac{M_0}{N_0}\,. \tag{3.1}$$

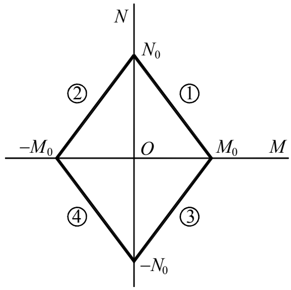

Fig 2. Linear yield conditions

In the case of two forces F_1, F_2 it should be written four inequalities of yield condition (3.1) for each locus apex $j=1,2,3,4$:

$$\varphi_{v,j} = \boldsymbol{M}_{0v} - \boldsymbol{\Phi}_v \boldsymbol{S}_{v,j} \geq \mathbf{0}\,,$$
$$\boldsymbol{S}_{v,j} = \left(M_{ev,j} + M_{rv},\, N_{ev,j} + N_{rv}\right)^{\mathrm{T}}, \qquad \boldsymbol{\Phi}_v = \begin{bmatrix} 1 & c_k \\ -1 & c_k \\ 1 & -c_k \\ -1 & -c_k \end{bmatrix}, \tag{3.2}$$
$$v = 1,2,\ldots,\zeta\,,\;\; j = 1,2,\ldots,p\,.$$

Yield conditions for whole structure read

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\boldsymbol{S}_j \geq \mathbf{0}\,, \qquad \boldsymbol{S}_j = \left(\boldsymbol{M}_{e,j} + \boldsymbol{M}_r,\, \boldsymbol{N}_{e,j} + \boldsymbol{N}_r\right)^{\mathrm{T}},\;\; j \in J\,. \tag{3.3}$$

Elastic forces $\boldsymbol{S}_{ec}$ resulted by load combination, monotonically increasing loading or distortion can be included into yield (3.2) as follow:

$$\boldsymbol{\varphi}_{v,j} = \boldsymbol{M}_{0v} - \boldsymbol{\Phi}_v\left(\boldsymbol{S}_{v,j} + \boldsymbol{S}_{ec}\right) \geq \mathbf{0}\,, \qquad v = 1,2,\ldots,\zeta\,,\quad j = 1,2,\ldots,p\,. \tag{3.4}$$

Mathematical models of frame analysis problem are presented in Table 2.

Table 2. Formulations of frame analysis problems

Static formulation	Kinematic formulation
find $\min \mathscr{F}'(\boldsymbol{S}_r) = \min \ \frac{1}{2}\boldsymbol{S}_r^{\mathrm{T}}\boldsymbol{D}\boldsymbol{S}_r$, (3.5) subject to $\boldsymbol{A}\boldsymbol{S}_r = \boldsymbol{0}$, (3.6) $\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\boldsymbol{S}_j \geq \boldsymbol{0}$, (3.7) $\boldsymbol{S}_j = \boldsymbol{S}_{ej} + \boldsymbol{S}_r$ for all $j \in J$.	find $\max \mathscr{F}''(\boldsymbol{S}_r, \boldsymbol{u}_r, \boldsymbol{\lambda}_j) = \max\left\{-\frac{1}{2}\boldsymbol{S}_r^{\mathrm{T}}\boldsymbol{D}\boldsymbol{S}_r - \sum_{j=1}^{p}\boldsymbol{\lambda}_j^{\mathrm{T}}\left(\boldsymbol{M}_0 - \boldsymbol{\Phi}\boldsymbol{S}_{ej}\right)\right\}$ (3.8) subject to $\boldsymbol{D}\boldsymbol{S}_r + \sum_{j=1}^{p}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\lambda}_j - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_r = \boldsymbol{0}$, (3.9) $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$. (3.10)

Optimal solution $\boldsymbol{S}_r^*$ of quadratic problem (3.5) – (3.7) is unique and corresponds to the level of dissipated energy $D_{\max}$. Optimal solution of kinematic analysis problem formulation (3.8) – (3.10) is $\boldsymbol{S}_r^*$, $\boldsymbol{u}_r^*$, $\boldsymbol{\lambda}_j^*$, also $\boldsymbol{\Theta}_p^* = \boldsymbol{\Phi}^{\mathrm{T}} \sum_{j=1}^{p}\boldsymbol{\lambda}_j^*$ and magnitude $D_{\max} = \sum_{j=1}^{p}\boldsymbol{\lambda}_j^{*\mathrm{T}}\boldsymbol{M}_0$, $j \in J$.

Problems of residual displacement variation bound determination. Plastic strains $\boldsymbol{\Theta}_{pv}$ occur in the section v when complementary slackness conditions of mathematical programming are satisfied:

$$\boldsymbol{\lambda}_{v,j}^{\mathrm{T}}\left(\boldsymbol{M}_{0v} - \boldsymbol{\Phi}_v\boldsymbol{S}_{j,v}\right) = 0 \quad (\text{or } \boldsymbol{\lambda}_{v,j}^{\mathrm{T}}\,\boldsymbol{\varphi}_{v,j} = 0), \quad \boldsymbol{\lambda}_{v,j} \geq \boldsymbol{0}, \ v \in V, \ j \in J . \qquad (3.11)$$

Yield condition satisfied as an equality $\varphi = 0$ can become an inequality $\varphi < 0$ during future deformation process but plasticity multiplier remains $\lambda > 0$. Such behaviour of the structure can not be evaluated because of complementary slackness conditions $\boldsymbol{\lambda}_{v,j}^{\mathrm{T}}\,\boldsymbol{\varphi}_{v,j} = 0$, $v \in V$, $j \in J$.

Vectors of displacement bounds $\overline{\boldsymbol{u}}_{r,inf}$, $\overline{\boldsymbol{u}}_{r,sup}$ are obtained analysing all possible loading histories $\boldsymbol{F}(t)$. Meanwhile vectors $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$ are such approximate, safe bounds of residual displacement $\boldsymbol{u}_{r,inf} \leq \overline{\boldsymbol{u}}_{r,inf}^*$, $\overline{\boldsymbol{u}}_{r,sup}^* \leq \boldsymbol{u}_{r,sup}$. Often there are restricted only residual displacements in optimal design problems, then stiffness constraints (2.28) can be rewritten as follow:

$$\boldsymbol{u}_{r,\min} \leq \boldsymbol{u}_{r,inf} , \qquad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,\max} . \qquad (3.12)$$

The first problem. Components $\widetilde{u}_{ri,inf}$, $\widetilde{u}_{ri,sup}$ $i=1,2,...,m$ of residual displacement variation bound vectors $\widetilde{\boldsymbol{u}}_{r,inf}$, $\widetilde{\boldsymbol{u}}_{r,sup}$ are obtained by solving the following linear mathematical programming problem: find

$$\begin{matrix}\max\\ \min\end{matrix}\ \boldsymbol{H}_i^* \widetilde{\boldsymbol{\lambda}} = \begin{bmatrix}\widetilde{u}_{ri,sup}\\ \widetilde{u}_{ri,inf}\end{bmatrix}, \quad i=1,2,...,m\ , \tag{3.13}$$

subject to

$$\boldsymbol{B}_\lambda^* \widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_r \boldsymbol{S}_r^*, \quad \widetilde{\boldsymbol{\lambda}} \geq \boldsymbol{0}\ , \tag{3.14}$$

$$\widetilde{\boldsymbol{\lambda}}^T \widetilde{\boldsymbol{M}}_0 \leq D_{\max}\ . \tag{3.15}$$

This mathematical model represents fictitious structure, i.e. system having displacements $\widetilde{\boldsymbol{u}}_{r,inf}$, $\widetilde{\boldsymbol{u}}_{r,sup}$, which “envelope” displacements $\boldsymbol{u}_r$ of given structure at shakedown. Vector $\boldsymbol{S}_r^*$ and magnitude of $D_{\max}$ are found according to optimal solutions of the problem (3.8) – (3.10); $\widetilde{\boldsymbol{M}}_0$ is a vector of limit moments of fictitious structure:

$$\widetilde{M}_{0v} = \max \boldsymbol{\Phi}_v \left(\boldsymbol{S}_r^* + \boldsymbol{S}_{ev,j}\right) \geq \boldsymbol{0}\ , \quad v \in V\ , \quad j \in J\ . \tag{3.16}$$

Elastic forces $\boldsymbol{S}_e^*$ and matrix $\boldsymbol{\Phi}^*$ of such linear yield conditions $\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi} \boldsymbol{S}_j \geq \boldsymbol{0}$, which satisfy condition (3.16), are determined together with vector $\widetilde{\boldsymbol{M}}_0$. Then following equality is valid: $\widetilde{\boldsymbol{M}}_0 = \boldsymbol{\Phi}^* \left(\boldsymbol{S}_r^* + \boldsymbol{S}_e^*\right)$.

The second problem. Components $u_{ri,inf}$, $u_{ri,sup}$ $i=1,2,...,m$ of displacement variation bound vectors $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$ are obtained, due basic solution vectors of $\boldsymbol{\lambda}_0 \geq \boldsymbol{0}$ of strain compatibility equations $\boldsymbol{B}_\lambda^* \boldsymbol{\lambda}_0 = \boldsymbol{B}_r \boldsymbol{S}_r^*$. Basic variables $\boldsymbol{\lambda}_0' \geq \boldsymbol{0}$ compounding vector $\boldsymbol{\lambda}_0 \geq \boldsymbol{0}$, can be determined according to the formula $\boldsymbol{\lambda}_0' = \left(\boldsymbol{B}_\lambda'^*\right)^{-1} \boldsymbol{B}_r \boldsymbol{S}_r^*$. Let there would be $\boldsymbol{\lambda}_{0,\,\omega} \geq \boldsymbol{0}$ satisfying conditions (3.15), set of subscripts $\omega = 1,2,...,\omega_\lambda$ is Ω. Vectors of residual displacements $\boldsymbol{u}_{r0,\,\omega}$ are calculated according to the formula:

$$\boldsymbol{u}_{r0,\,\omega} = \boldsymbol{H}^* \boldsymbol{\lambda}_{0\,\omega}\ , \quad \omega \in \Omega\ . \tag{3.17}$$

Vectors $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$ ($\widetilde{\boldsymbol{u}}_{r,inf} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \widetilde{\boldsymbol{u}}_{r,sup}$) are constructed by picking components of all vectors $\boldsymbol{u}_{r0,\omega}$ ($\omega \in \Omega$) with maximal and minimal values.

Frame volume minimization at shakedown. Mathematical model of the problem ($\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, σ_{yk} , L_k , $k \in K$ are known) reads: find

$$\min \sum_{k} L_k A_k \ , \tag{3.18}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0}\,, \tag{3.19}$$

$$\sum_{j=1}^{p} \boldsymbol{\lambda}_j^{\mathrm{T}} \left[\boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right)\right] = 0\,, \quad \boldsymbol{\lambda}_j \geq \boldsymbol{0}\,, \quad \boldsymbol{\lambda} = \sum_{j=1}^{p} \boldsymbol{\lambda}_j\,, \quad j \in J\,, \tag{3.20}$$

$$A_k \geq A_{k,\min}\,, \quad k \in K\,, \tag{3.21}$$

$$\boldsymbol{u}_{r,\min} \leq \boldsymbol{u}_{r,inf}\,, \qquad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,\max}\,. \tag{3.22}$$

In the problem (3.18) – (3.22) unknowns are the cross–sectional areas A_k , $k \in K$ of frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ ($M_{0k} = \sigma_{yk} W_{pl,k} = \xi\left(\sigma_{yk}, A_k\right)$, $N_{0k} = \sigma_{yk} A_k$ are functions of the cross–sectional area A_k and material yield limit σ_{yk}). Lower bounds of the cross–sectional areas $A_{k,\min}$ are included into constructive constraints (3.21). It is not difficult to introduce elastic displacements into stiffness constraints (3.22) (see inequalities (2.28)). Limit moments $\boldsymbol{M}_0$, influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are related with design variables A_k , $k \in K$; listed matrixes are recalculated during the solution process of the problem (3.18) – (3.22).

From solution algorithm scheme (Fig 3) it is possible to see that in the beginning problem (3.18) – (3.22) is solved when stiffness conditions (3.22) are changed into constraints corresponding to the holonomic deformation process:

$$\boldsymbol{u}_{r,\min} \leq \boldsymbol{H}\,\boldsymbol{\lambda} \leq \boldsymbol{u}_{r,\max}\,. \tag{3.23}$$

After optimal solution of the problem (3.18) – (3.21), (3.23) is found, stricter stiffness constraints (3.22) are verified using displacement bounds $\tilde{\boldsymbol{u}}_{r,inf}$, $\tilde{\boldsymbol{u}}_{r,sup}$ or $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$. Numerical illustration is presented for frame shown in Fig 4a.

Mathematical models of frame load optimization, presented in the section 3, are constructed according to (2.29) – (2.33) and have the following objective function: $\max \left\{ \boldsymbol{T}_{sup}^{\mathrm{T}} \boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^{\mathrm{T}} \boldsymbol{F}_{inf} - \sum_{j=1}^{p} \boldsymbol{\lambda}_j^{\mathrm{T}} \left[\boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\overline{\boldsymbol{G}}\,\boldsymbol{\Theta}_p + \boldsymbol{S}_{ej}\right)\right] \right\}$.

Here unknowns are $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ and $\boldsymbol{\lambda}_j$, $j \in J$.

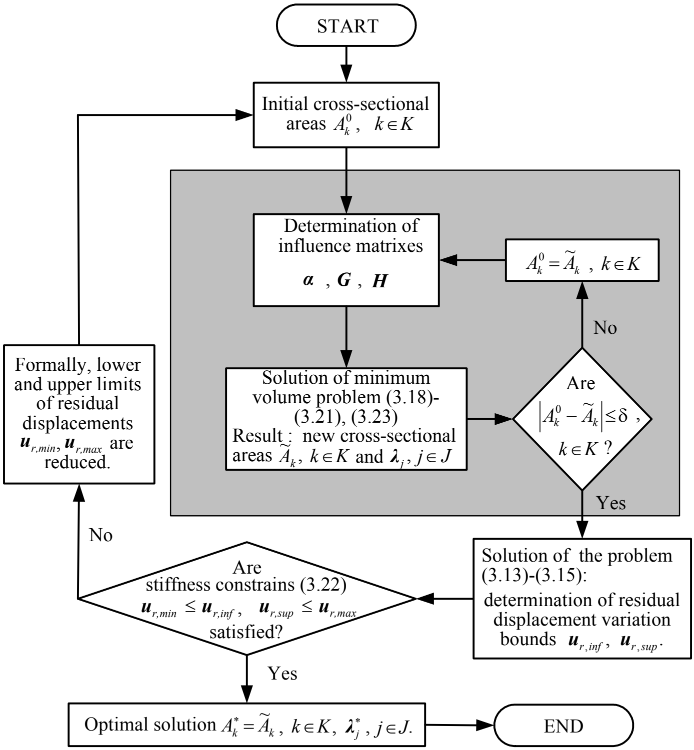

Fig 3. Flowchart of the proposed solution algorithm

4. Optimal design of trusses

Evaluation of bar stability. For trusses stability conditions (besides strength and stiffness requirements) are related with recommendations of EC3, when admissible forces of compressive bars are obtained by reduction of their material yield limit σ_y (vector of limit forces $\boldsymbol{N}_0$ ($N_{0j} = \sigma_{yk} A_k$, $k \in K$) is substituted by $\boldsymbol{N}_{0,cr}$). Then yield conditions of discretized truss read:

$$\boldsymbol{\varphi}_{\max} = \boldsymbol{N}_0 - \boldsymbol{N}_r - \boldsymbol{N}_{e,\max} \ge \boldsymbol{0} \, , \; \boldsymbol{\varphi}_{\min} = \boldsymbol{N}_{0,cr} + \boldsymbol{N}_r + \boldsymbol{N}_{e,\min} \ge \boldsymbol{0} \, . \qquad (4.1)$$

Here $\boldsymbol{N}_{e,\max} = \boldsymbol{\alpha}_{sup} \boldsymbol{F}_{sup} + \boldsymbol{\alpha}_{inf} \boldsymbol{F}_{inf}$, $\boldsymbol{N}_{e,\min} = \boldsymbol{\alpha}_{sup} \boldsymbol{F}_{inf} + \boldsymbol{\alpha}_{inf} \boldsymbol{F}_{sup}$ are vectors of minimal and maximal values of elastic forces; $\boldsymbol{N}_0 = (N_{0k})^{\mathrm{T}}$, $\boldsymbol{N}_{0,cr} = (N_{0k,cr})^{\mathrm{T}}$,

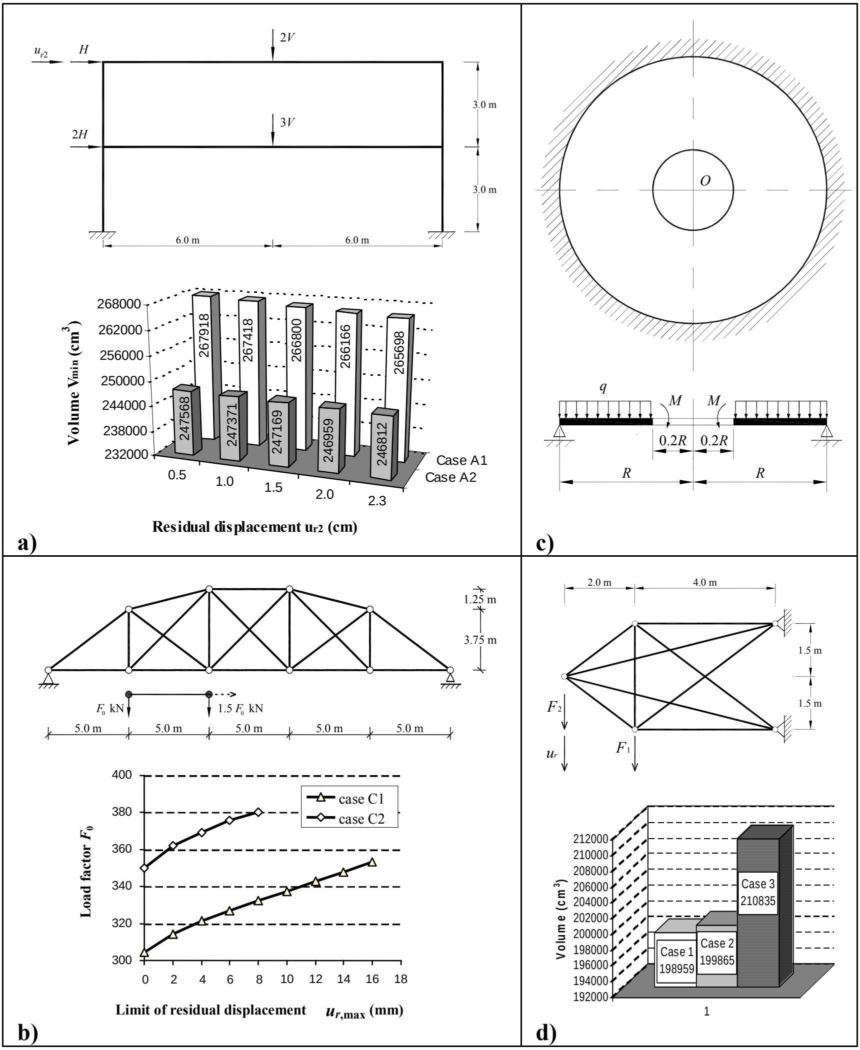

Fig 4. Objects of numerical examples

$N_{0,k} = \sigma_{yk} A_k$, $N_{0,k,cr} = \varphi_k \sigma_{yk} A_k$, $k \in K$. $N_{0,cr,k}$ are calculated according to the formulas:

$$N_{0,cr,k} = \varphi_k N_{0,k}, \qquad \varphi_k = \frac{1}{\Phi_k + \left[\Phi_k^2 - \bar{\lambda}_k^2\right]^{0.5}}, \tag{4.2}$$

when

$$\Phi_k = 0.5\left(1 + a\left(\overline{\lambda}_k - 0.2\right) - \overline{\lambda}_k^2\right),\ \ \overline{\lambda}_k = \frac{\lambda_k}{\lambda_{1k}}\sqrt{\beta_A} = \frac{\lambda_k}{\pi[E_k / \sigma_{y,k}]^{0.5}}\sqrt{\beta_A}\ ,\ \ k \in K\ .$$

Here E_k is an elasticity modulus of the *k*–th bar; $\lambda_k = L_k / i_k$ is bar slenderness, where i_k is the radius of gyration of the *k*–th bar. In the case of bar under pure compression $\beta_A = 1$; value of imperfection factor a , į depends on the shape of cross–sections and properties of applied material. Possible failure because of stability lost is not evaluated when $\boldsymbol{N}_{0,cr} = \boldsymbol{N}_0$.

The problem of truss volume minimization. Project of minimum volume of adapted truss is determined (when load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, material yield limit σ_{yk} and lengths L_k of all k ($k \in K$) elements are prescribed) by solving the following problem: *find truss of minimum volume* $V = \sum_k L_k A_k$ *(* $k \in K$ *), satisfying requirements of strength, stiffness and stability*.

Mathematical model of non–linear problem reads: find

$$\min \sum_k L_k A_k\ , \tag{4.3}$$

subject to

$$\boldsymbol{\varphi}_{\max}(A) = \boldsymbol{N}_0 - \boldsymbol{G\Theta}_p - \boldsymbol{N}_{e,\max} \geq \boldsymbol{0}\ ,\ \boldsymbol{\varphi}_{\min}(A) = \boldsymbol{N}_{0,cr} + \boldsymbol{G\Theta}_p + \boldsymbol{N}_{e,\min} \geq \boldsymbol{0}\ , \tag{4.4}$$

$$\boldsymbol{N}_0 = \left(N_{0,k}\right)^{\mathrm{T}}\ ,\ \boldsymbol{N}_{0,cr} = \left(N_{0,k,cr}\right)^{\mathrm{T}}\ ,\ N_{0,k} = \sigma_{yk} A_k\ ,\ N_{0,k,cr} = \varphi_k \sigma_{yk} A_k\ , \tag{4.5}$$

$$A_k \geq A_{k,\min}\ ,\quad k \in K\ , \tag{4.6}$$

$$\boldsymbol{\Theta}_p = \boldsymbol{\lambda}_{\max} - \boldsymbol{\lambda}_{cr}\ , \tag{4.7}$$

$$\boldsymbol{\lambda}_{\max}^{\mathrm{T}} \boldsymbol{\varphi}_{\max} = 0\ ,\ \ \boldsymbol{\lambda}_{cr}^{\mathrm{T}} \boldsymbol{\varphi}_{\min} = 0\ ,\ \ \boldsymbol{\lambda}_{\max} \geq \boldsymbol{0}\ ,\ \ \boldsymbol{\lambda}_{cr} \geq \boldsymbol{0}\ , \tag{4.8}$$

$$\boldsymbol{u}_{r,\min} \leq \boldsymbol{u}_{r,inf}\ ,\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,\max}\ . \tag{4.9}$$

Unknowns are cross–sectional areas A_k , $k \in K$ of bars and vectors of plastic multipliers $\boldsymbol{\lambda}_{\max}$, $\boldsymbol{\lambda}_{cr}$. Stiffness constraints (4.9) are realised via restriction of nodal displacements. Influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on design variable A_k , $k \in K$. Possibility to evaluate load combinations, change of temperature and distortions makes mathematical model (4.4) – (4.9) important for practical design.

Load optimization problem. In mathematical model presented in dissertation, limit axial forces $\boldsymbol{N}_0$, $\boldsymbol{N}_{0,cr}$ and limits of residual displacements

$\boldsymbol{u}_{r,\min}$, $\boldsymbol{u}_{r,\max}$ are prescribed. Optimal solution is $\boldsymbol{F}^*_{sup}$, $\boldsymbol{F}^*_{inf}$ and $\boldsymbol{\lambda}^*_{\max}$, $\boldsymbol{\lambda}^*_{cr}$. Numerical examples are presented for 9–bar and 20–bar (in case of moving load) trusses (see Fig 4b and Fig 4d).

5. Incremental structure analysis for unloading phenomenon fixation at shakedown

Sequentially load variation field is extended in line with an arbitrary increment not changing its form. Then mathematical model of incremental analysis problem reads: find

$$\min \ \frac{1}{2}\sum_k \left(\boldsymbol{S}_{r\Sigma,k} + \Delta\boldsymbol{S}^z_{rk}\right)^{\mathrm{T}} \boldsymbol{D}_k \left(\boldsymbol{S}_{r\Sigma,k} + \Delta\boldsymbol{S}^z_{rk}\right), \tag{5.1}$$

subject to

$$\sum_k \boldsymbol{A}_k \Delta\boldsymbol{S}^z_{rk} = \boldsymbol{0}, \ \ \Delta\boldsymbol{S}^z_{rk} = \left(\Delta\boldsymbol{S}^z_{rk1}, \Delta\boldsymbol{S}^z_{rk2}, ..., \Delta\boldsymbol{S}^z_{rkl}, ..., \Delta\boldsymbol{S}^z_{rk\eta_k}\right)^{\mathrm{T}}, \tag{5.2}$$

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\boldsymbol{S}^z_{c,kl,j} + \Delta\boldsymbol{S}^z_{rkl}\right) \ge 0, \ \ C_k = \left(S_{0k}\right)^2, \ \ k \in K,$$

$$\boldsymbol{S}^z_{c,kl,j} = \boldsymbol{S}^z_{e,kl,j} + \boldsymbol{S}_{r\Sigma,kl} = \sum_z \Delta\boldsymbol{S}^{z-1}_{e,kl,j} + \Delta\boldsymbol{S}^z_{e,kl,j} + \sum_z \Delta\boldsymbol{S}^{z-1}_{r,kl}, \ l \in L, \ j \in J. \tag{5.3}$$

The problem (5.1) – (5.3) optimal solution is vector of residual force increments $\Delta\boldsymbol{S}^{*z}_{rk}$ at the end of z–th stage. For this problem Kuhn–Tucker conditions read:

$$\boldsymbol{D}\Delta\boldsymbol{S}^{*z}_r + \boldsymbol{D}\boldsymbol{S}_{r\Sigma} + \nabla\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\lambda}^{*z} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}^{*z}_r = \boldsymbol{0}, \quad \boldsymbol{\lambda}^{*z\mathrm{T}}\boldsymbol{\varphi} = 0, \quad \boldsymbol{\lambda}^{*z} \ge \boldsymbol{0}, \ \ z \in Z.$$

Plastic strain at the end of the z–th stage are calculated taking into account all apexes $j \in J$ of elastic force locus:

$$\boldsymbol{\Theta}^{*z}_{pkl} = \sum_j \left[\nabla\varphi_{kl,j}\,(\boldsymbol{S}^z_{c,kl,j} + \Delta\boldsymbol{S}^{*z}_{rkl})\right]^{\mathrm{T}} \lambda^{*z}_{kl,j}, \qquad k \in K, \ l \in L, \ j \in J. \tag{5.4}$$

At the end of loading process, when $z = \tau$, residual forces $\boldsymbol{S}^*_r = \sum_{z=1}^{\tau} \Delta\boldsymbol{S}^{*z}_r$, displacements $\boldsymbol{u}^*_r$ and plastic strains $\boldsymbol{\Theta}^*_p$ are obtained.

Modified mathematical model. The mathematical model (5.1) – (5.3) can be transformed by applying residual force influence matrix $\overline{\boldsymbol{G}}$ and plastic strains $\boldsymbol{\Theta}_{p\Sigma}$, at the beginning of the z–th stage. Then residual forces $\boldsymbol{S}_{r\Sigma} = \sum_z \Delta\boldsymbol{S}^{z-1}_r$ are calculated by applying formula: $\boldsymbol{S}_{r\Sigma} = \overline{\boldsymbol{G}}\boldsymbol{\Theta}^{z-1}_p$. Expression of the objective function is rewritten as follows:

$$\frac{1}{2}\,\boldsymbol{\Theta}_{p\Sigma}^{\mathrm{T}}\overline{\boldsymbol{G}}^{\,\mathrm{T}}\boldsymbol{D}\boldsymbol{G}\boldsymbol{\Theta}_{p\Sigma}+\frac{1}{2}\left(\Delta\boldsymbol{S}_r^z\right)^{\mathrm{T}}\boldsymbol{D}\Delta\boldsymbol{S}_r^z+\left(\Delta\boldsymbol{S}_r^z\right)^{\mathrm{T}}\boldsymbol{D}\overline{\boldsymbol{G}}\,\boldsymbol{\Theta}_{p\Sigma} \tag{5.5}$$

The first member of the expression (5.5) is constant $\frac{1}{2}\,\boldsymbol{\Theta}_{p\Sigma}^{T}\boldsymbol{G}^{T}\boldsymbol{D}\boldsymbol{G}\boldsymbol{\Theta}_{p\Sigma}=const$ and does not influence the optimal solution $\Delta\boldsymbol{S}_r^{*z}$ of the problem (5.1) – (5.3). Mathematical model of incremental analysis (5.1) – (5.3) obtains the form: find

$$\min\left(\frac{1}{2}\Delta\boldsymbol{S}_r^{z\,\mathrm{T}}\boldsymbol{D}\Delta\boldsymbol{S}_r^z+\Delta\boldsymbol{S}_r^{z\,\mathrm{T}}\boldsymbol{D}\overline{\boldsymbol{G}}\,\boldsymbol{\Theta}_{p\Sigma}\right), \tag{5.6}$$

subject to (5.2) and (5.3).

Optimal solution $\Delta\boldsymbol{S}_r^{*z}$ satisfies Rosen optimality criterion (Kuhn–Tucker conditions):

$$\boldsymbol{D}\Delta\boldsymbol{S}_r^{*z}+\boldsymbol{D}\overline{\boldsymbol{G}}\,\boldsymbol{\Theta}_{p\Sigma}+\nabla\boldsymbol{\varphi}^{\,\mathrm{T}}\boldsymbol{\lambda}^{*z}-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{u}_r^{*z}=\boldsymbol{0}\,,\ \boldsymbol{\lambda}^{*z\mathrm{T}}\boldsymbol{\varphi}=0\,,\quad \boldsymbol{\lambda}^{*z}\geq\boldsymbol{0}\,. \tag{5.7}$$

About cross–section unloading phenomenon. In the section 2 defined requirements for unloading phenomenon

$$\lambda_i>0\,,\qquad \varphi_i=C_i-f_i\left(\boldsymbol{S}_i\right)\geq 0\,,\qquad i=1,2,...,\zeta,\ \ i\in I\,. \tag{5.8}$$

are taken into account by verifying the sign of plastic multiplier increments at each calculation stage:

$$\Delta\lambda_i^z=\lambda_i^z-\lambda_i^{z-1}\,. \tag{5.9}$$

When $\Delta\lambda_i^z<0$, unloading phenomenon is developing in the i –th cross–section. This moment is fixed and analysis problem solution is continued in order to be sure that unloading phenomenon really appeared in the cross–section, i.e. conditions (5.8) are satisfied. Having fixed that, fictitious plasticity constant is introduced and calculation process is continued.

Incremental analysis of bending plate. Mathematical model of static plate analysis problem formulation is obtained from the problem (5.2) – (5.3),(5.6): find

$$\min\left\{\frac{1}{2}\ \Delta\boldsymbol{M}_r^{z\mathrm{T}}\boldsymbol{D}\Delta\boldsymbol{M}_r^z+\Delta\boldsymbol{M}_r^{z\mathrm{T}}\boldsymbol{D}\overline{\boldsymbol{G}}\,\boldsymbol{\Theta}_{p\Sigma}\right\}, \tag{5.10}$$

subject to

$$\boldsymbol{A}\Delta\boldsymbol{M}_r^z=\boldsymbol{0}\,,\ \Delta\boldsymbol{M}_{rk}^z=\left(\Delta\boldsymbol{M}_{rk1}^z,\Delta\boldsymbol{M}_{rk2}^z,...,\Delta\boldsymbol{M}_{rkl}^z,...,\Delta\boldsymbol{M}_{rk\eta_k}^z\right)^{\mathrm{T}}, \tag{5.11}$$

$$\varphi_{kl,j}=C_k-\boldsymbol{M}_{kl,j}^{z\mathrm{T}}\boldsymbol{\Phi}\boldsymbol{M}_{kl,j}^z\geq 0\,,\ \ C_k=\left(M_{0k}\right)^2,\ \ \boldsymbol{M}_{kl,j}^z=\boldsymbol{M}_{c,kl,j}^z+\Delta\boldsymbol{M}_{r,kl}^z\,,$$

$$\boldsymbol{M}_{c,kl,j}^{z} = \boldsymbol{M}_{e,kl,j}^{z} + \boldsymbol{M}_{r\Sigma,kl} = \sum_{z} \Delta \boldsymbol{M}_{e,kl,j}^{z-1} + \Delta \boldsymbol{M}_{e,kl,j}^{z} + \sum_{z} \Delta \boldsymbol{M}_{r,kl}^{z-1} ,$$

$$k \in K , \; l \in L , \; j \in J . \qquad (5.12)$$

Unknowns of the problem (5.10) – (5.12) are increments of residual moments $\Delta \boldsymbol{M}_{r}^{z}$. After optimal solution $\Delta \boldsymbol{M}_{r}^{*z}$ determination of each z–th loading stage, the plastic deformations $\boldsymbol{\Theta}_{p}^{*z}$ and residual displacements $\boldsymbol{u}_{r}^{*z}$ are calculated.

Numerical examples are presented for annular plate (Fig 4c).

Results and conclusions

1) The features of recent shakedown theory development and achievements in field of structural optimization are shown. 2) New potential, which is provided by connections between mathematical programming and extremum energy principles, are shown for formulation of analysis and optimization problems of shakedown theory and their numerical solution. 3) It is proved that residual displacements are varying non–monotonically. Dual problems of mathematical programming can not be applied for analysis of stress–strain state of unloading system. 4) New ways of elastic force application in the conditions of stress state admissibility are proposed; compatibility between continuum and discrete object of deformable body mechanics is detailed. 5) It is determined that Haar–Kármán principle is suitable only for systems with holonomic behaviour. 6) New non–linear mathematical models of load oriented and optimum design problems with constraints ensuring shakedown and serviceability of structure are created. 7) It is proved that complementary slackness condition simulates only holonomic deformation process during shakedown of structure. Problem of adapted structure optimization is not a classical mathematical programming problem. It is difficult to use complementary slackness conditions when loading history is unknown. 8) Constructed mathematical models are universal: when stiffness constraints are neglected optimal solution is obtained according to cyclic–plastic failure, it is very easy to interpret monotonically increasing loading. Type of cyclic–plastic collapse is identified using complementary slackness conditions. 9) The possibility to apply strain compatibility equations for formulation of new mathematical models of residual displacement variation bound calculation problems is proved. Different from other authors, who use global conditions for displacement restriction, in this work, the possibility of using local restriction of displacements is created. 10) On the basis of general mathematical models of analysis and optimization problems, the optimization technique of adapted bar–structures is implemented: new non–linear mathematical models for frames (under strength and stiffness constraints) and trusses (with stability conditions (EC3)) optimization problems, solution algorithms and numerical experiments are carried out. 11) Different from other authors, who treat moving load as a separate type of load, in this dissertation such load is treated

as a separate case of repeated load. It is proved that in such case locus of elastic forces is non–symmetric. Optimization technique is created for trusses subjected by moving load. 12) An algorithm for optimization problem solution which involves holonomic elastic–plastic strain development process and allows taking into account possible unloading phenomenon appearance at cross–sections is proposed. The main part of the algorithm is common for both load oriented and optimum design problems. 13) Software for non–linear problem solution (it is based on Rosen project gradient method) is created. Rosen optimality criterion (Kuhn–Tucker conditions) ensures compatibility of residual strains and makes possible calculation of plastic strain and residual displacement increments without solving dual analysis problems (it is a basis for incremental analysis). 14) Technique of incremental analysis for unloading phenomenon fixation is created; numerical experiment is carried out for bending plate with non–linear von Mises yield conditions.

A List of Publications on the Subject of the Dissertation

1. Atkočiūnas J., Jarmolajeva E., Merkevičiūtė D. Optimal shakedown loading for circular plates. *Structural and Multidisciplinary Optimization*, Vol 27, No 3. Berlin: Springer–Verlag, 2004, p. 178–188. ISSN 1615–147X. (ISI)
2. Skaržauskas V., Merkevičiūtė D., Atkočiūnas J. Optimisation des portiques dans les conditions d'adaptation avec des restrictions en déplacements. *Revue Européenne de Génie Civil*, Vol 9, No 4. Paris: Hermes–Lavoisier, 2005, p. 435–453. ISSN 1279–5119.
3. Merkevičiūtė D., Atkočiūnas J. Minimum volume of trusses at shakedown – mathematical models and new solution algorithms. *Mechanics* (Mechanika), No 2 (52). Kaunas: Technologija, 2005, p. 47–54. ISSN 1392–1207.
4. Atkočiūnas J., Merkevičiūtė D., Rimkus L. Dual formulations of structure analysis problem at shakedown. Their application limits. *Journal of Civil Enginiering and Management*, Vol IX, Supplement 2. Vilnius: Technika, 2003, p. 91–99 (in Lithuanian). ISSN 1392–3730.
5. Merkevičiūtė D., Atkočiūnas J. Incremental method for unloading phenomenon fixation at shakedown. *Journal of Civil Engineering and Management*, Vol IX, No 3. Vilnius: Technika, 2003, p. 178–191. ISSN 1392–3730.
6. Skaržauskas V., Merkevičiūtė D., Atkočiūnas J. Optimization of elastic–plastic frames at shakedown. *Statyba* (Civil Engineering), Vol VII, No 6. Vilnius: Technika, 2001, p. 427–434 (in Lithuanian). ISSN 1392–1525.
7. Atkočiūnas J., Merkevičiūtė D. Optimal shakedown design of bar systems: strength, stiffness and stability constraints. In: *Proceedings of the 7th International Conference on Computational Structures Technology*, September 7–9, 2004, Lisbon, Portugal (Eds. B. H. V. Topping, C. A. Mota Soares), Civil–Comp Press, Stirling, Scotland, 2004, p. 361–362 and in CD (19 p.). ISBN 0–948749–95–4.
8. Merkevičiūtė D., Atkočiūnas J. Minimum volume of trusses under shakedown loading. In: *Proceedings of 12th Polish–Ukrainian Conference on Theoretical Foundations of Civil Engineering*, June 28–July 2, 2004, Warsaw, Poland

(Ed. W. Szczesniak). Warsaw University of Technology, Warsaw, Poland, 2004, p. 901–909. ISBN 5–7763–8880–5.
9. Merkevičiūtė D., Atkočiūnas J. Optimal shakedown design of trusses: strength, stiffness and stability constraints. In: *Proceedings of 8th International Conference on Modern Building Materials, Structures and Techniques*, May 19–21, 2004, Vilnius, Lithuania (Eds. Zavadskas E. K., Vainiūnas P., Mazzolani F. M.). Vilnius Gediminas Technical University,Vilnius: Technika, 2004, p. 830–839. ISSN 9986–05–757– 4. (ISI proceedings)
10. Merkevičiūtė D., Atkočiūnas J. Method for unloading phenomenon fixation at shakedown. In: *Proceedings of 3rd International Conference on Strength, Durability and Stability of Materials and Structures SDSMS'03*, September 17–19, 2003, Klaipėda, Lithuania (Eds. I. Pritykin, F. Shmidt, J. Janutėnienė, V. Leišis). Kaunas University of Technology, Kaunas: Technologija, 2003, p. 189–200. ISBN 9955–09–549–0.
11. Atkočiūnas J., Merkevičiūtė D. Kuhn–Tucker conditions and load optimization problem at shakedown. In: *Proceedings of 1st CEACM Conference on Computational Mechanics, 15th International Conference "CMM–2003 – Computer Methods in Mechanics"*, June 3–6, 2003, Gliwice/Wisla, Poland (Eds. T. Burczyński, P. Fedeliński, E. Majchrzak). Silesian Technical University, Gliwice, Poland, 2003, 9 p. ISBN 83–914632–4–9 (CD–ROM).
12. Merkevičiūtė D. Holonomic analysis principles in nonlinear problems of structures in shakedown. In: *Proceedings of the 6th Conference of Lithuanian Young Scientists "Lithuania without science – Lithuania without future"*, March 27–28, 2003, Vilnius, Lithuania (6–osios Lietuvos jaunųjų mokslininkų konferencijos „Lietuva be mokslo – Lietuva be ateities", įvykusios Vilniuje 2003 m. kovo 27–28 d., medžiaga. Statyba). Vilnius: Technika, 2003. p. 225–230. ISBN 9986–05–618–7.
13. Merkevičiūtė D. Optimization of structures at shakedown applying Rosen and random search algorithms. In: *Proceedings of the 5th Conference of Lithuanian Young Scientists "Lithuania without science – Lithuania without future"*, March 27–29, 2002, Vilnius, Lithuania (5–osios Lietuvos jaunųjų mokslininkų konferencijos „Lietuva be mokslo – Lietuva be ateities", įvykusios Vilniuje 2002 m. kovo 27–29 d., medžiaga. Statyba). Vilnius: Technika, 2002, p. 326–332 (in Lithuanian). ISBN 9986–05–559–8.

Briefly about the author

Dovilė Merkevičiūtė was born in Vilnius on August 18, 1977.

BSc and MSc in Civil Engineering, Faculty of Civil Engineering, Vilnius Gediminas Technical University in 1999 and 2001, respectively. Civil engineering studies at Politecnico di Torino, Italy according to Socrates/Erasmus program in 2000–2001. PhD studies at Vilnius Gediminas Technical University in 2001–2005. Assistant at the Structural Mechanics Department, Vilnius Gediminas Technical University in 2001–2002 and 2004–2005. Young science employee and assistant at the Structural Mechanics Department, Vilnius Gediminas Technical University, since September 2005.

TAMPRIŲ-PLASTINIŲ PRISITAIKANČIŲ SISTEMŲ OPTIMIZACIJA SU STANDUMO IR STABILUMO SĄLYGOMIS

Bendroji darbo charakteristika

Tyrimų sritis ir aktualumas. Statybinės mechanikos optimizavimo uždaviniai, nagrinėjami šiame darbe, yra įžanginis konstrukcijų optimalaus projektavimo etapas, pagrįstas kietojo deformuojamo kūno mechanikos principais, matematinio programavimo teorija, jos metodais ir jų mechanine interpretacija. Norint skaičiavimą pritaikyti realioms konstrukcijos darbo sąlygoms, būtina optimizavimo uždavinių matematiniuose modeliuose kuo tiksliau įvertinti konstrukcijos medžiagos savybes, išorinius poveikius ir kitus veiksnius. Iš dalies tai pasiekiama, įvertinus medžiagos plastines savybes, kuriomis pasižymi nemaža statybinių konstrukcijų, ypač metalinių. Konstrukcijų skaičiavimas ir projektavimas, įvertinant plastines deformacijas, leidžia efektyviau išnaudoti jų laikomąją galią ir sudaryti ekonomiškesnius projektus (disertaciniame darbe tyrimai plėtojami vadovaujantis idealaus plastiškumo teorija). Kita vertus, realūs konstrukcijos poveikiai dažniausiai yra cikliški – šiame darbe įvertinamas ir kartotinės kintamosios apkrovos pobūdis. Apkrova šiame darbe laikoma kvazistatine, ji nusakoma ne konkrečia apkrovimo istorija, o tik viršutinėmis ir apatinėmis jėgų kitimo ribomis (darbe nagrinėjama deterministinė optimizavimo uždavinių formuluotė).

Kai apkrova kartotinė, konstrukcija gali netekti savo laikomosios galios, t.y. patirti ciklinę plastiškąją suirtį dėl susikaupusių per didelių plastinių deformacijų arba mažaciklio nuovargio. Tačiau jeigu apkrovimo pradžioje dėl plastinio tekėjimo atsiradusios liekamosios įrąžos kartu su kintamąja įrąžų dalimi niekur neišeina už leistinųjų ribų, konstrukcija *prisitaiko* prie duotosios apkrovos ir toliau dirba tik tampriai. Šį reiškinį nagrinėja prisitaikomumo teorija. Taikant klasikines prisitaikomumo teorijos Melano ir Koiterio teoremas, elementariais metodais galima analizuoti tik nesudėtingų sistemų prisitaikymą. Statybos inžinerijai aktualus bet kokio sudėtingumo kartotinai apkraunamų tampriųjų–plastinių prisitaikančių konstrukcijų skaičiavimas šiuolaikiniais skaitiniais metodais. Tai lėmė ir šios disertacijos mokslinių tyrimų krypties ir metodų pasirinkimą, būtent *tampriųjų-plastinių sistemų, veikiamų kintamosios kartotinės apkrovos, optimizacija su standumo ir stabilumo sąlygomis, taikant ekstreminius energinius principus, matematinio programavimo teoriją, skaitinius metodus ir naudojant šiuolaikines kompiuterines technologijas*. Prisitaikančioms konstrukcijoms skaičiuoti skirtų mokslo darbų gausėjimas taip pat rodo šių tyrimų svarbą. Tačiau čia itin maža darbų, skirtų prisitaikančioms konstrukcijoms su standumo ir stabilumo sąlygomis optimizuoti. Prisitaikančių konstrukcijų optimizavimo uždavinius spręsti sudėtinga dar ir todėl, kad disipacinių sistemų įtempių ir deformacijų būvis priklauso nuo apkrovimo istorijos. Disertacijoje sprendžiami ne tik bendrieji prisitaikančių konstrukcijų analizės ir optimizavimo teorijos dalykai, pateikti ir nauji strypinių konstrukcijų optimizavimo metodai ir algoritmai, turintys statybos inžinerijoje ir praktinę reikšmę.

Pagrindinis darbo tikslas. Tampriųjų-plastinių prisitaikančių sistemų optimizacijos teorijos tolesnė plėtotė, naujų skaičiavimo metodų ir algoritmų kūrimas. Skaitinių eksperimentų realizavimas.

Darbo uždaviniai: 1) atlikti prisitaikančių sistemų šiuolaikinių skaičiavimo metodų analizę; 2) suformuluoti idealiai tampriųjų-plastinių prisitaikančių diskretinių sistemų ekstreminius energinius principus; 3) sudaryti analizės ir optimizacijos netiesinių uždavinių bendruosius matematinius modelius; 4) išryškinti prisitaikančių sistemų ekstreminių optimizacijos uždavinių ir netiesinio matematinio programavimo teorijos sąsajas; 5) atlikti strypinių konstrukcijų optimizaciją, įvertinant jų standumo ir stabilumo reikalavimus pagal antrosios ribinių būvių grupės nuorodas: projektinio ir tikrinamojo uždavinių matematiniai modeliai, sprendimo metodai, algoritmai, skaitiniai eksperimentai; 6) sukurti prisitaikiusios konstrukcijos optimizavimo uždavinių, esant neišsamiai pradinei informacijai, sprendimo algoritmą; 7) sukurti skerspjūvių nusikrovimo inkrementinės analizės metodiką.

Mokslinis naujumas. 1) Atskleistos matematinio programavimo, plačiai paplitusio ekstreminių uždavinių sprendimo metodo ir ekstreminių energinių principų sąsajų teikiamos naujos galimybės, formuluojant prisitaikomumo teorijos analizės ir optimizavimo uždavinius ir juos skaitiškai sprendžiant. 2) Sudaryti nauji prisitaikančių sistemų tikrinamojo ir projektinio optimizacijos netiesinių uždavinių matematiniai modeliai su užtikrinančiomis prisitaikomumą ir tinkamumą eksploatuoti sąlygomis-apribojimais. 3) Nustatyta, kad Haar-Kármán principas tinka tik holonominės elgsenos (nenusikraunančioms prisitaikymo metu) sistemoms. 4) Įrodyta, kad liekamieji poslinkiai prisitaikymo metu kinta nemonotoniškai – nusikraunančių sistemų įtempių ir deformacijų būvio analizei netaikytini dualieji matematinio programavimo uždaviniai. 5) Įrodyta, kad matematinio programavimo griežtumo sąlyga modeliuoja tik holonominį deformavimo procesą konstrukcijos prisitaikymo metu. 6) Parodyta liekamųjų deformacijų darnos lygčių taikymo galimybė, formuluojant naujus liekamųjų poslinkių kitimo ribų skaičiavimo uždavinių matematinius modelius. 7) Realizuota prisitaikančių strypinių konstrukcijų optimizavimo metodika: sudaryti nauji rėmų, atsižvelgiant į jų stiprumo ir standumo, o santvarų – dar ir į stabilumo (EN3) sąlygas, uždavinių netiesiniai matematiniai modeliai, sprendimo algoritmai, atlikti skaitiniai eksperimentai. 8) Sukurta metodika santvaroms, veikiamoms judamosios apkrovos, optimizuoti (projektinis ir tikrinamasis aspektai). 9) Sukurtas prisitaikančių konstrukcijų nusikrovimo fiksavimo inkrementinis metodas, atliktas lenkiamosios plokštės su Mizeso netiesine takumo sąlyga eksperimentinis skaičiavimas.

Darbo aprobacija ir publikacijos. Pagrindiniai disertacinio darbo tyrimo rezultatai aptarti dešimtyje mokslinių techninių konferencijų. Disertacijos tema publikuota trylika mokslo darbų, iš jų šeši – recenzuojamuose periodiniuose mokslo leidiniuose, penki – tarptautinių ir du – respublikinių konferencijų pranešimų leidiniuose.

Darbo apimtis. Lietuvių kalba parašytą disertaciją sudaro įvadas, penki atskiri skyriai, išvados, literatūros ir disertacijos autorės publikacijų sąrašai. Disertacijos tekstas pateiktas 131 puslapyje, tekste yra 36 paveikslai, 10 lentelių.

Disertacijos turinys

Prisitaikančių sistemų skaičiavimo metodų apžvalgai skirtas pirmasis skyrius, svarbi jo dalis yra lyginamoji prisitaikomumo būvio poslinkių skaičiavimo metodų analizė. Darbe rezultatai pateikti laikantis tokio principo: iš pradžių pateikiami optimizacijos uždavinių bendrieji matematiniai modeliai (2 skyrius), vėliau jie taikomi prisitaikiusių rėmų (3 skyrius) ir santvarų (4 skyrius) optimizacijai, atsižvelgiant į kiekvienos konstrukcijos specifiką, tačiau išlaikant metodologinę vienovę.

Antrajame skyriuje sudarytieji analizės ir optimizacijos uždavinių matematiniai modeliai remiasi kietojo deformuojamo kūno mechanikos ekstreminiais energiniais principais. Tuo atveju matematinio programavimo teorija padėjo formuluoti sudėtingų plastiškumo teorijos konstrukcijų prisitaikomumo uždavinių matematinius modelius ir juos išspręsti skaitiniais metodais. Prisitaikiusios prie ciklinės apkrovos konstrukcijos liekamosios įrąžos, poslinkiai ir deformacijos, nenagrinėjant apkrovimo istorijos, randami sprendžiant dualiąsias – statinę ir kinematinę – analizės uždavinio formuluotes. Strypinės sistemos pavyzdžiu parodyta, kad tikruosius liekamuosius poslinkius pagal dualųjį matematinio programavimo uždavinį galima apskaičiuoti tik tuo atveju, kai prisitaikymo metu konstrukcijos skerspjūviai nusikrauna (tada ir liekamieji poslinkiai prisitaikymo metu kinta monotoniškai). Antrasis skyrius baigiamas optimizacijos uždavinių matematinių modelių analize, kuria parodyta, kad šie uždaviniai nėra klasikinė netiesinio matematinio programavimo problema: standumo sąlygos suponuoja ne visos pradinės informacijos apribojimą.

Trečiasis ir ketvirtasis skyrius skirti rėmams ir santvaroms optimizuoti. Matematinio programavimo griežtumo sąlyga neleidžia įvertinti skerspjūvių nusikrovimo reiškinio: liekamųjų poslinkių kitimo riboms skaičiuoti disertacijoje sukurtas fiktyviosios konstrukcijos metodas, jo pagrindu sudarytas tiesinio programavimo uždavinys. Jo sprendimo rezultatai įtraukiami į pagrindinio optimizacijos uždavinio sąlygas. Sudaryti nauji rėmų, atsižvelgiant į stiprumo ir standumo bei santvarų, atsižvelgiant dar ir į stabilumo (EN3) sąlygas, uždavinių netiesiniai matematiniai modeliai, sprendimo algoritmai, atlikti detalūs skaitiniai eksperimentai. Ketvirtajame skyriuje sukurta metodika santvarų, veikiamų judamosios apkrovos optimizacijai.

Prisitaikančių sistemų nusikrovimo inkrementinei analizei skirtas penktasis skyrius – čia vėl grįžtama prie bet kokio sudėtingumo konstrukcijos analizės galimybės, skaitiniam eksperimentui pasitelkus žiedinę lenkiamąją plokštę su Mizeso takumo sąlyga.

Pagrindiniai darbo rezultatai ir išvados

1. Atskleisti šiuolaikinės prisitaikomumo teorijos plėtros bruožai ir įdirbis konstrukcijų optimizavimo srityje.

2. Išryškintos matematinio programavimo ir ekstreminių energinių principų sąsajų teikiamos naujos galimybės, formuluojant prisitaikomumo teorijos analizės ir optimizavimo uždavinius ir juos skaitiškai sprendžiant.
3. Įrodyta, kad liekamieji poslinkiai prisitaikymo metu bendruoju atveju kinta nemonotoniškai – nusikraunančių sistemų įtempių ir deformacijų būvio analizei netaikytini dualieji matematinio programavimo uždaviniai.
4. Pasiūlyti nauji tampriųjų įrąžų panaudojimo įtempių būvio leistinumo (takumo) sąlygose variantai, detalizuotas kietojo deformuojamo kūno mechanikos kontinualių ir diskretinių objektų suderinamumas (idealios formos skerspjūvis).
5. Nustatyta, kad Haar-Kármán principas tinka tik holonominės elgsenos sistemoms.
6. Sudaryti nauji prisitaikančių sistemų tikrinamojo ir projektinio optimizacijos netiesinių uždavinių matematiniai modeliai su užtikrinančiomis prisitaikymą ir tinkamumą eksploatuoti sąlygomis-apribojimais.
7. Įrodyta, kad matematinio programavimo griežtumo sąlyga modeliuoja tik holonominį deformavimo procesą – prisitaikiusios konstrukcijos optimizavimo uždavinys nėra klasikinis matematinio programavimo uždavinys. Sunku naudotis griežtumo sąlygomis, kai apkrovimo istorija nežinoma arba nenagrinėjama.
8. Sudarytieji uždavinių matematiniai modeliai universalūs: atsisakius standumo (dažniausiai įlinkių) apribojimų, gaunami optimalūs sprendiniai pagal ciklinę plastiškąją suirtį, labai paprasta interpretuoti vienkartę apkrovą. Ciklinės plastiškosios suirties tipas identifikuojamas, pasitelkus matematinio programavimo griežtumo sąlygą.
9. Įrodyta liekamųjų deformacijų darnos lygčių taikymo galimybė, formuluojant naujus liekamųjų poslinkių kitimo ribų skaičiavimo uždavinių matematinius modelius (ribų uždavinys yra sudėtinė optimizacijos uždavinio dalis). Skirtingai nuo kitų autorių, naudojančių globalinę poslinkių ribojimo sąlygą, darbe sukurta lokalinė poslinkių ribojimo galimybė.
10. Bendrųjų analizės ir optimizavimo uždavinių matematinių modelių pagrindu realizuota prisitaikančių strypinių konstrukcijų optimizavimo metodika: sudaryti nauji rėmų, atsižvelgiant į jų stiprumo ir standumo, o santvarų – dar ir į stabilumo (EN3) sąlygas, uždavinių netiesiniai matematiniai modeliai, sprendimo algoritmai, atlikti skaitiniai eksperimentai.
11. Judamoji apkrova disertaciniame darbe, skirtingai nuo kitų autorių, išskiriančių ją į atskirą apkrovų rūšį, traktuojama tik kaip atskiras kartotinės apkrovos atvejis. Įrodyta, kad tampraus įrąžų sprendimo hodografas tuo atveju būna nesimetriškas. Sukurta santvarų, veikiamų judamosios apkrovos, optimizavimo metodika.
12. Sudarytas optimizacijos uždavinių sprendimo algoritmas, apimantis tiek holonominį tampriųjų-plastinių deformacijų vystymosi procesą, tiek leidžiantis atsižvelgti į galimą skerspjūvių nusikrovimą. Pagrindinė algoritmo dalis yra bendra ir patikrinamajam, ir projektiniam uždaviniui.

13. Sukurta netiesinių uždavinių sprendimo programa (Rozeno algoritmas). Rozeno optimalumo kriterijus (Kuno ir Takerio sąlygos) užtikrina liekamųjų deformacijų darną ir leidžia skaičiuoti plastinių deformacijų bei liekamųjų poslinkių prieaugius, nesprendžiant analizės dualiųjų matematinio programavimo uždavinių – pagrindas inkrementinei prisitaikomumo analizei.
14. Sukurtas prisitaikančių konstrukcijų nusikrovimo fiksavimo inkrementinis metodas, atliktas lenkiamosios plokštės su Mizeso netiesine takumo sąlyga eksperimentinis skaičiavimas.

Konstrukcijų, patiriančių kintamus kartotinius mechaninius ir kitus poveikius, stiprumo, standumo ir stabilumo problemų sprendimas glaudžiai susietas su prisitaikomumo teorija. Šią teoriją veiksmingai naudoti dar ir dabar trukdo nepakankama matematinė kai kurių uždavinių formalizacija ir nenuoseklus matematinio programavimo teorijos ir jos metodų panaudojimas. Disertaciniame darbe sukurta bendra tampriųjų-plastinių prisitaikančių sistemų optimizacijos su standumo ir stabilumo sąlygomis, taikant ekstreminius energinius principus, matematinio programavimo teoriją, skaitinius metodus ir šiuolaikines kompiuterines technologijas, metodika. Visos darbo teorinės išvados ir pasiūlytos metodikos iliustruotos skaitiniais pavyzdžiais.

Trumpos žinios apie autorių

Dovilė Merkevičiūtė gimė 1977 m. rugpjūčio 18 d. Vilniuje.

1999 m. įgijo statybos inžinerijos bakalauro laipsnį Vilniaus Gedimino technikos universiteto Statybos fakultete. 2000–2001 m. studijavo Turino technikos universitete, Italijoje. 2001 m. įgijo statybos inžinerijos mokslo magistro laipsnį Vilniaus Gedimino technikos universiteto Statybos fakultete. 2001–2005 m. – Vilniaus Gedimino technikos universiteto doktorantė. 2001–2002 m. ir 2004–2005 m. dirbo asistente Vilniaus Gedimino technikos universiteto Statybinės mechanikos katedroje. Nuo 2005 m. rugsėjo dirba jaunesniąja mokslo darbuotoja ir asistente Vilniaus Gedimino technikos universiteto Statybinės mechanikos katedroje.

Dovilė Merkevičiūtė
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Dovilė Merkevičiūtė
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Spausdino UAB „Biznio mašinų kompanija“, Gedimino pr. 60, LT-01110 Vilnius