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OPTIMIZATION OF BENDING PLATES AT SHAKEDOWN

Summary of Doctoral Dissertation
Technological Sciences, Civil Engineering (02T)

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VILNIUS GEDIMINAS TECHNICAL UNIVERSITY

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General Characteristics of the Dissertation

Topicality of the problem. Mathematics and mechanics always had influence on each other's development, rising new scientific ideas and implementing them. One of the examples of such close their relation is wide application of mathematical programming in solving extreme mechanical problems. This theory also enables problem formulation of adapted structures and their realization. Using mathematical programming not only new optimization technique of bending plates at shakedown is developed, but also relation between Kuhn-Tucker conditions and strain compatibility (Saint-Venant) equations and dependences of associative yield law of the deformable body mechanics is showed in the dissertation. That is why investigations were done in functional space as well as using discrete models of structures (deterministic formulation of problems was considered).

Calculation of the structures, taking in to account plastic strains and effect of variable repeated loads, allows to relate optimization problems of structural mechanics with practical design process. Adapted perfectly elastic-plastic structure satisfies strength conditions and it is safe with respect to cyclic-plastic collapse. But it can do not satisfy its serviceability requirements, for instance, stiffness ones. Therefore, not only strength, but also stiffness conditions-constraints should be included in the discrete mathematical models of bending plate parameter or load variation bound optimization problems (exactly such problems are considered in the dissertation). Mathematical models of nonlinear problems are constructed applying method of equilibrium elements and are solved by iterations using Rosen project gradient algorithm. The feature of this research work is that the theory of mathematical programming accompanies investigation of optimization problem from the construction of the mathematical model up to its numerical solution, at the same time revealing mechanical meaning optimality criterion of applied Rosen algorithm. Then, in the case of convex nonlinear programming (analogical to the case of linear programming), the values of dual variables are determined at once. That quickens realization of adapted structure optimization problems, especially having in the mind, that stress-strain state of the dissipative systems depends on loading history.

The main objective. To develop application of mathematical programming in the theory of solid deformable body mechanics: to show that for extreme problem formulated on the basis of Castigliano's principle Kuhn-Tucker conditions are strain compatibility equations. To create new methods and effective algorithms for solving parameter and load distribution optimization problems of adapted elastic-plastic bending plate under deflection constraints.

The research object. Kuhn-Tucker conditions in the extreme problems of elasticity and plasticity theory. Mathematical models of optimization problems (design and verification aspect) of perfectly elastic-plastic isotropic plate and their solution algorithms.

Methodology of research. Extreme energy principles of deformable body mechanics and theory of mathematical programming are used. Discrete mathematical models of optimization problems are constructed on the basis of the equilibrium finite element method. Investigations are performed and results of numerical experiments are obtained according to assumptions of small displacements.

Main tasks: 1) to reveal relation between Kuhn-Tucker conditions and the main equations of adapted systems; 2) to make analysis of current calculation methods of adapted structures including bending plates; 3) to construct general mathematical models of plate analysis and optimization problems; 4) to determine peculiarities of bending plate residual deflection variation and construct mathematical model for their bound calculation problem; 5) to perform shakedown load variation bound optimization of bending plates; 6) to optimize distribution of bending plate parameters taking into account strength and stiffness conditions-constraints; 7) to create algorithms to solve optimization problems with non-linear Hubert-Mises yield conditions.

Scientific novelty: 1) In the dissertation, new potential of mathematical programming theory, known as solution method of extreme problems, is revealed for problem formulation in shakedown theory and their numerical solution. 2) The role of Kuhn-Tucker conditions is discovered in formulating general equations of deformable body mechanics via stresses (strain compatibility equations of elastic-plastic structures). It is shown that in the problems of limit equilibrium Kuhn-Tucker conditions also contain dependences of associated yield law. 3) Discovering mechanical meaning of optimality criterion of Rosen project gradient method enables to obtain dual solution of non-linear structure analysis problem at once. 4) Improved mathematical model for deflection variation bound calculation of elastic-plastic bending plate enables to state stiffness conditions more exactly in the optimization problems of plates at shakedown. At the same time the influence of mathematical programming slackness conditions, which do not allow taking in to account physical unloading phenomenon, is reduced. 5) New mathematical models of adapted plate optimization problems (design and verification aspect) are created.

Approbation and publications. The main results of the dissertation were submitted in 9 scientific conferences. Eleven scientific papers were published on the topic of dissertation: 1 of them was published in the scientific journal included into the master list of Institute for Scientific Information (ISI Master List), 1 – in the proceedings of conference organized by international scientific organizations (ISI Proceedings), 3 – in the reviewed scientific periodicals included the databases from approved list by Lithuanian Scientific Council in 2006.

The scope of the scientific work. The thesis written in Lithuanian consists of introduction, four main chapters, conclusions, a list of references (182 positions) and a list of dissertation author's publications (11 positions). The total scope of the dissertation is 103 pages, 14 pictures and 12 tables.

1. The Main Calculation Principles of Structures at Shakedown

In the first chapter, development of the elastic-plastic body shakedown theory is explored and recent tendency of its progress is described, talking about the adapted structure optimization problem in detail.

Review of investigations and objective of the dissertation are presented in the first subsection. Growing number of scientific works dedicated to adapted structure calculation and optimization shows importance of researches in this field. Applying the classical Melan and Koiter theorems, it is possible to analyse shakedown of only simple systems by elementary methods. Meanwhile for civil engineering, applying recent numerical methods, calculation of any complexity elastic-plastic structures, including bending plates, subjected by variable repeated load is relevant. The following tendency of shakedown theory current development showed up:

- *extension of classical theorems for realistic material constitutive law, including hardening;*
- *evaluation of cyclic temperature variation, dynamic loads;*
- *behavior analysis of geometrically non-linear structures at shakedown;*
- *restriction of residual displacements and strains at the sections of optimal structure at shakedown.*

The first studies about optimal design of structures at shakedown showed up in 1958. But only during the last three decades it was started to use the duality theory of mathematical programming in the shakedown theory. Optimal design with constrained deflections is difficult problem of optimal operating and there is still small number of works concerning this field. In the dissertation it was achieved even more: here mathematical programming is not only the tool for numerical solving of shakedown optimization problems, but also instrument for constructing of problem mathematical models using energy principles.

Generalized variables of structure shakedown problems are described in the second subsection. The method of equilibrium finite elements is used for structure discretization. Stress state is defined by vector of internal forces $\boldsymbol{S}=\left(\boldsymbol{S}_1,\boldsymbol{S}_2,...,\boldsymbol{S}_\zeta\right)^T$ ($\zeta \le s\times v$, where s is the number of finite elements ($k=1,2,...,s$, $k\in K$) and v is the number of element nodes (sections), $l=1,2,...,v$, $l\in L$). Strain vector $\boldsymbol{\Theta}=(\boldsymbol{\Theta}_1,\boldsymbol{\Theta}_2,...,\boldsymbol{\Theta}_\varsigma)^T$ is matching force vector $\boldsymbol{S}$. Displacements $\boldsymbol{u}$ and load $\boldsymbol{F}$ (load variation bounds are $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$) compound the other couple of dual variables. Variation field of elastic forces $\boldsymbol{S}_e$, taking into account all possible load combinations (their number is $p=2^m$), is restricted by convex and symmetrical, in respect to its own centre, polyhedron $\boldsymbol{S}_e(t)=\boldsymbol{\alpha}\,\boldsymbol{F}(t)$ with apexes $\boldsymbol{S}_{ej}$ ($j=1,2,...,p$):

$$\boldsymbol{S}_{ej} = \boldsymbol{\alpha}\,\boldsymbol{F}_j \;\; (\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}). \tag{1.1}$$

Plasticity constant C of elastic-plastic structure relates to dimensions and material, i.e. limit force S_0, of discrete model. In this research it is assumed that limit force S_{0k} ($k \in K$) is constant in whole finite element. Then yield conditions

$$\varphi_{kl,j} = C_k - f_{kl,j}\left(\boldsymbol{S}_{ekl,j} + \boldsymbol{S}_{rkl}\right) \ge 0\,, \quad k \in K\,, \quad l \in L\,, \quad j \in J \tag{1.2}$$

are verified in each nodal point l of the k-th finite element for all load combinations $j \in J$. Statically admissible residual forces $\boldsymbol{S}_r$ satisfy equilibrium equations $\boldsymbol{A}\boldsymbol{S}_r = \boldsymbol{0}$ and yield conditions (1.2).

Kinematically possible residual displacements $\boldsymbol{u}_r$ satisfy geometrical equations:

$$\boldsymbol{A}^T \boldsymbol{u}_r = \boldsymbol{D}\boldsymbol{S}_r + \boldsymbol{\Theta}_p\,. \tag{1.3}$$

Here $\boldsymbol{D}$ is flexibility matrix, $\boldsymbol{\Theta}_p = \left(\boldsymbol{\Theta}_{pkl}\right)^T$ is vector of plastic strains:

$$\boldsymbol{\Theta}_{pkl} = \sum_j \left[\nabla\varphi_{kl,j}\,(\boldsymbol{S}_{ekl,j} + \boldsymbol{S}_{rkl})\right]^T \lambda_{kl,j}\,, \quad \lambda_{kl,j} \ge 0\,; \quad k \in K\,, \quad l \in L\,, \quad j \in J\,. \tag{1.4}$$

Residual strains $\boldsymbol{\Theta}_r = \boldsymbol{D}\boldsymbol{S}_r + \boldsymbol{\Theta}_p$ and displacements $\boldsymbol{u}_r$ of structure at shakedown can be non-unique: they depend on the particular loading history $\boldsymbol{F}(t)$, ($\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$), which was implemented in order to reach shakedown state. Thus, if load is defined only by variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$, the calculation of exact values of residual displacements becomes problematical, it is possible to determine only their lower $\boldsymbol{u}_{r,\,inf}$ and upper $\boldsymbol{u}_{r,\,sup}$ variation bounds ($\boldsymbol{u}_{r,\,inf} \le \boldsymbol{u}_r(t) \le \boldsymbol{u}_{r,\,sup}$).

Extreme energy principles and improvement of general mathematical models conclude the first chapter. Here general mathematical models of cyclic-plastic failure problems and with them related extreme energy principles are described briefly, following shakedown analysis problem (principle of minimum complementary energy of deformation):

find
$$\min \frac{1}{2}\boldsymbol{S}_r^T \boldsymbol{D}\boldsymbol{S}_r \tag{1.5}$$

subject to
$$\boldsymbol{A}\boldsymbol{S}_r = \boldsymbol{0}\,, \tag{1.6}$$

$$\boldsymbol{f}\left(\boldsymbol{S}_r + \boldsymbol{S}_{ej}\right) \le \boldsymbol{C}\,, \quad j \in J \tag{1.7}$$

is examined. Here vectors $\boldsymbol{C}$ and $\boldsymbol{S}_{ej}$ are known. The global solution of analysis problem $\boldsymbol{S}_r^*$ is connecting-link between stress and strain state describing equations and dependences in optimization problems.

In the dissertation, general mathematical models are constructed with non-linear yield conditions for their future orientation to Hubert-Mises criterion applied to plates (Table 1).

Table 1. Mathematical models of shakedown state optimization problems

Verification problem	Optimal design problem
find	find
$\max\left(\boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf} + \boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup}\right)$ (1.8)	$\min \ \boldsymbol{L}^T \boldsymbol{S}_0$ (1.14)
subject to	subject to
$\min \frac{1}{2} \boldsymbol{S}_r^T \boldsymbol{D} \boldsymbol{S}_r$ (1.9)	$\min \frac{1}{2} \boldsymbol{S}_r^T \boldsymbol{D} \boldsymbol{S}_r$ (1.15)
$\boldsymbol{A} \boldsymbol{S}_r = \boldsymbol{0}$, (1.10)	$\boldsymbol{A} \boldsymbol{S}_r = \boldsymbol{0}$, (1.16)
$\boldsymbol{\varphi}_j = \boldsymbol{C} - \boldsymbol{f}_j\left(\boldsymbol{S}_r + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0}$, $j \in J$ (1.11)	$\boldsymbol{\varphi}_j = \boldsymbol{C} - \boldsymbol{f}_j\left(\boldsymbol{S}_r + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0}$, $j \in J$; (1.17)
$\boldsymbol{F}_{sup} \geq \boldsymbol{0}$, $\boldsymbol{F}_{inf} \geq \boldsymbol{0}$, (1.12)	$\boldsymbol{C} = \xi\left(\boldsymbol{S}_0\right)$, $\boldsymbol{S}_0 \geq \boldsymbol{0}$,
$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$. (1.13)	$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$. (1.18)

The main difficulty for effective mathematical model creation of adapted structure optimization problems is evaluation of loading history and with it related unloading phenomenon. The loading history is taken into account by introducing extreme values of elastic forces $\boldsymbol{S}_{ej}$, $j \in J$ and displacements $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$.

2. Kuhn-Tucker Conditions Composing the Main Equations of Structures at Shakedown

The shakedown theory incorporates the main equations and dependences of elastic and plastic state of deformable body mechanics and plasticity. In the second chapter, applying the theory of mathematical programming, those equations and dependences are investigated according to the Castigliano's principle (analysis problem (1.5)–(1.7)):

find
$$\min \mathscr{F}'(\boldsymbol{x}) \tag{2.1}$$

subject to
$$g_i(\boldsymbol{x}) \leq 0, \quad i = 1, 2, \ldots, \zeta, \tag{2.2}$$
$$h_i(\boldsymbol{x}) = 0, \quad i = 1, 2, \ldots, m. \tag{2.3}$$

For convex functions Kuhn-Tucker conditions are sufficiency criterion of initial global solution: $\boldsymbol{x}^*$ *is optimal solution if such scalar multipliers* λ_i $(i = 1, 2, \ldots, \zeta)$ and u_i $(i = 1, 2, \ldots, m)$ *exist that*

$$\nabla \mathscr{F}'\left(\boldsymbol{x}^*\right) + \nabla \boldsymbol{g}^T\left(\boldsymbol{x}^*\right)\boldsymbol{\lambda} + \nabla \boldsymbol{h}^T\left(\boldsymbol{x}^*\right)\boldsymbol{u} = \boldsymbol{0}, \tag{2.4}$$

$$\boldsymbol{\lambda}^T \boldsymbol{g}\left(\boldsymbol{x}^*\right) = 0, \tag{2.5}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}. \tag{2.6}$$

Components of vectors $\boldsymbol{\lambda}$ and $\boldsymbol{u}$ are Lagrange multipliers, equalities $\lambda_i \, g_i(\boldsymbol{x}^*) = 0$, $\lambda_i \geq 0$, $i = 1, 2, \ldots, \zeta$ are complementary slackness conditions of mathematical programming.

In the second subsection of the chapter, equations of elasticity theory are considered in the context of Kuhn-Tucker conditions. Solution of elasticity theory equations by means of stresses is associated with finding six components of stress vector $\boldsymbol{\sigma} = \left(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}\right)^T$, satisfying three differential equations of body equilibrium $\mathcal{A}\,\boldsymbol{\sigma} = \mathbf{0} \quad \in V$ and six Beltrami-Michell dependences

$$\begin{cases} (1+\nu)\nabla^2\sigma_{11} + \dfrac{\partial^2}{\partial x_1^2}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (1+\nu)\nabla^2\sigma_{23} + \dfrac{\partial^2}{\partial x_2 \partial x_3}(\sigma_{11}+\sigma_{22}+\sigma_{33}) = 0 \end{cases} \quad \text{in volume } V \tag{2.7}$$

and boundary conditions. Here ∇^2 is Laplacian, ν is Poisson's ratio.

A direct integration of the nine differential equation system with partial fluxions, in respect of six stress functions to be found, is difficult. Therefore, often Castigliano's variation principle, which allows to achieve identical satisfaction of six Saint-Venant's strain compatibility equations, is applied for solving the boundary elasticity problem in terms of stresses. But the Castigliano's variation principle does not regulate the number of independent strain compatibility equations. Thus, it remains unclear whether the elasticity problem equation system in terms of stresses is over-defined or not. In the dissertation, three independent strain compatibility equations are obtained for modified boundary conditions.

Optimality criterion of Rosen algorithm and Kuhn-Tucker conditions are identical. For structures in limit state, Rosen criterion represents dependences of associative plastic yield law and compatibility equations of plastic strain velocities. For optimization problems of adapted structures, Rosen criterion is equations of force or displacement method of structural mechanics. That depends on which extreme principle (Castigliano or Lagrange) was applied for considered problem mathematical model construction.

Investigation of strain compatibility equations of elastic and elastic-plastic structures by means of united – energy and mathematical programming – principles is new methodical view in calculation of structures at shakedown.

3. Shakedown Load Optimization of Plates under Stiffness Constraints

In the third chapter, plate shakedown load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, satisfying optimality criterion max $\left\{\boldsymbol{T}_{sup}^{T}\boldsymbol{F}_{sup}+\boldsymbol{T}_{inf}^{T}\boldsymbol{F}_{inf}\right\}$, strength and stiffness conditions of plate, are to be found. Weight factors $\boldsymbol{T}_{sup}$, $\boldsymbol{T}_{inf}$ of optimality criterion show the influence of separate load components to general objective function.

Shakedown load optimization problem couples two different problems of structure mechanics. The first problem is determination of statically admissible residual bending moments of plate, which satisfy principle of minimum complementary deformation energy (ensuring its strength, i.e. shakedown state). The second problem is verification of plate stiffness constraints. The solution of the first problem has influence to the results of the second one, i.e. calculation of residual displacement variation bounds, which are included into stiffness constraints, depends on it. Mathematical and mechanical interpretation of Rosen criterion is connecting-link between both problems.

In the second formulation of load optimization problem, equations and dependences of analysis problem are written according to full equation system of the plasticity theory.

The first formulation of the problem mathematical model. The shakedown state residual bending moments $\boldsymbol{M}_r^*$ and optimal load variation bounds $\boldsymbol{F}_{sup}^*$, $\boldsymbol{F}_{inf}^*$ are to be found by solving following problem:

find
$$\max\left\{\boldsymbol{T}_{sup}^{T}\boldsymbol{F}_{sup}+\boldsymbol{T}_{inf}^{T}\boldsymbol{F}_{inf}\right\}=W \qquad (3.1)$$

subject to
$$\left.\begin{aligned}&\min\frac{1}{2}\,\boldsymbol{M}_r^{''T}\tilde{\boldsymbol{D}}\boldsymbol{M}_r^{''}\\ &\varphi_{kl,j}=C_k-\boldsymbol{M}_{kl,j}^{T}\boldsymbol{\Phi}\boldsymbol{M}_{kl,j}\geq 0,\quad C_k=\left(M_{0k}\right),\\ &\boldsymbol{M}_{kl,j}=\boldsymbol{M}_{ekl,j}+\boldsymbol{M}_{rkl},\quad \boldsymbol{M}_r=\boldsymbol{B}^{T}\boldsymbol{M}_r^{''},\\ &\boldsymbol{F}_{inf}\geq\boldsymbol{0},\quad \boldsymbol{F}_{sup}\geq\boldsymbol{0},\end{aligned}\right\} \qquad (3.2)$$

and
$$\boldsymbol{u}_{r,min}\leq\boldsymbol{u}_r=\boldsymbol{H}\boldsymbol{\Theta}_p\leq\boldsymbol{u}_{r,max},\quad k\in K,\ l\in L,\ j\in J. \qquad (3.3)$$

Residual moments $\boldsymbol{M}_r^{''}$ are the force method variables obtained applying Jordan transformations: $\boldsymbol{M}_r=\left(\boldsymbol{M}_r^{'},\boldsymbol{M}_r^{''}\right)^T$, $\boldsymbol{M}_r=\boldsymbol{B}^{T}\boldsymbol{M}_r^{''}$; the matrix $\boldsymbol{B}=\left[\left(\boldsymbol{A}''\right)^T\left(\boldsymbol{A}'^{T}\right)^{-1},-\boldsymbol{I}\right]$ contains sub-matrices $\boldsymbol{A}'$, $\boldsymbol{A}''$ of equilibrium equation matrix $\boldsymbol{A}$. Then flexibility matrix of the plate is $\widetilde{\boldsymbol{D}} = \boldsymbol{BDB}^T$. Having solved analysis problem (3.2), plastic strains $\boldsymbol{\Theta}_p = \left(\boldsymbol{\Theta}_{pkl}\right)^T$, which are included into expressions of stiffness condition verification (3.3), are calculated as follows:

$$\boldsymbol{\Theta}_{pkl} = 2\sum_j \lambda_{kl,j}\boldsymbol{\Phi M}_{kl,j}\ ,\ \lambda_{kl,j}\left(C_k - \boldsymbol{M}_{kl,j}^T\ \boldsymbol{\Phi M}_{kl,j}\right) = 0\ ,$$

$$\lambda_{kl,j} \geq 0\ ,\ k \in K\ ,\ l \in L\ ,\ j \in J\ . \qquad (3.4)$$

Thus, unknowns of the problem (3.1)–(3.3) are not only $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, $\boldsymbol{M}_r^{''}$, but also plastic multipliers $\boldsymbol{\lambda}$ implicitly. Knowing vectors of multipliers $\boldsymbol{\lambda}_j$ ($j \in J$), residual deflections (or displacements) $\boldsymbol{u}_r$ are calculated according to formula $\boldsymbol{u}_r = \boldsymbol{H\lambda}$. Satisfaction of inequalities (3.3) $\boldsymbol{u}_{r,\,min} \leq \boldsymbol{H\,\Theta}_p \leq \boldsymbol{u}_{r,\,max}$ means that stiffness conditions of adapted plate can be not violated; $\boldsymbol{H}$ is influence matrix of residual deflections related to plastic strains, $\boldsymbol{u}_{r,\,min}$, $\boldsymbol{u}_{r,\,max}$ are vectors of admissible limits of residual displacement variation bounds (known in advance). $\boldsymbol{H} = \left(\boldsymbol{A\,K\,A}^T\right)^{-1}\boldsymbol{AK}$, where $\boldsymbol{K}$ is element stiffness matrix. Taking into account possible unloading phenomenon, inequality (3.3) should be transformed into the following ones:

$$u_{ri,\,min} \leq \min\ \boldsymbol{H}_i\boldsymbol{\lambda} = u_{ri,\,inf}\ ,\quad \max\ \boldsymbol{H}_i\boldsymbol{\lambda} = u_{ri,\,sup} \leq u_{ri,\,max}\ ,\quad i = 1,\,2,\,\ldots,m\ . \qquad (3.5)$$

Here the index i is related to the displacement vector $\boldsymbol{u}$ components $u_1, u_2, \ldots, u_m$. Bounds of the residual displacements $u_{ri,\,sup}$, $u_{ri,\,inf}$ are determined by solving the following problem of linear programming:

find
$$\begin{matrix}\max\\ \min\end{matrix}\ \widetilde{\boldsymbol{H}}_i\ \widetilde{\boldsymbol{\lambda}} = \begin{bmatrix} u_{ri,sup} \\ u_{ri,inf}\end{bmatrix} \qquad (3.6)$$

subject to
$$\boldsymbol{B}_\lambda\ \widetilde{\boldsymbol{\lambda}} = \boldsymbol{B}_r\boldsymbol{M}_r^*\ ,\quad \widetilde{\boldsymbol{\lambda}} \geq 0\ ,\quad \widetilde{\boldsymbol{\lambda}} = \left(\widetilde{\boldsymbol{\lambda}}_j\right),\quad \sum_j \widetilde{\boldsymbol{\lambda}}_j^T\ \widetilde{\boldsymbol{C}} \leq \widetilde{D}_{max}\ . \qquad (3.7)$$

Here $\widetilde{D}_{max}$ is maximum value of dissipated energy during shakedown process. Then displacements $\widetilde{\boldsymbol{u}}_{r,inf}$, $\widetilde{\boldsymbol{u}}_{r,sup}$ "envelop" the displacements $\boldsymbol{u}_r$ of shakedown process and state of the given structure.

The second formulation of the problem mathematical model. It is stated by applying full equation system of the plasticity theory and including complementary slackness conditions into objective function of optimization problem:

find $$\max\left\{\boldsymbol{T}_{sup}^{T}\boldsymbol{F}_{sup}+\boldsymbol{T}_{inf}^{T}\boldsymbol{F}_{inf}-\sum_{k}\sum_{l}\sum_{j}\lambda_{kl,j}\left(C_k-\boldsymbol{M}_{kl,j}^{T}\boldsymbol{\Phi}\boldsymbol{M}_{kl,j}\right)\right\}=W \quad (3.8)$$

subject to $\varphi_{kl,j}=C_k-\boldsymbol{M}_{kl,j}^{T}\boldsymbol{\Phi}\boldsymbol{M}_{kl,j}\geq 0$, $C_k=\left(M_{0k}\right)^2$, $\boldsymbol{M}_{kl,j}=\boldsymbol{M}_{ekl,j}+\boldsymbol{M}_{rkl}$, (3.9)

$$\boldsymbol{B}_{\theta}\boldsymbol{\Theta}_p=\boldsymbol{B}_r\boldsymbol{M}_r,\quad \boldsymbol{\Theta}_p=\left(\boldsymbol{\Theta}_{pkl}\right)^T,\quad \boldsymbol{\Theta}_{pkl}=2\sum_{j}\lambda_{kl,j}\boldsymbol{\Phi}\boldsymbol{M}_{kl,j},$$

$$\lambda_{kl,j}\geq 0,\quad k\in K,\quad l\in L,\quad j\in J; \quad (3.10)$$

$$\boldsymbol{F}_{inf}\geq\boldsymbol{0},\quad \boldsymbol{F}_{sup}\geq\boldsymbol{0},$$

and $$\boldsymbol{u}_{r,min}\leq\boldsymbol{H}\boldsymbol{\Theta}_p\leq\boldsymbol{u}_{r,max}. \quad (3.11)$$

Unknowns of the problem (3.8)–(3.11) are $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, $\boldsymbol{M}_r^{''}$, $\boldsymbol{\lambda}$. For the variable repeated loading it is not enough that „softer“ stiffness conditions (3.11) are satisfied. In order to evaluate possible unloading phenomenon, it is necessary to solve problem (3.6)–(3.7) analogically to the case of the problem (3.1)–(3.3).

Problem solution algorithm for implementation of both the first and the second problem formulations is described bellow. About solving of the first formulation problem (3.1)–(3.3) will be presented in detail. Solution of the optimization problem (3.1)–(3.3) is changed into the successive solution of two separate problems. Thus, calculation of optimal load variation bounds $\boldsymbol{F}_{sup}^{*}$, $\boldsymbol{F}_{inf}^{*}$ is performed step-by-step according to the cyclic-plastic collapse condition. The size of the step ν depends on admissible increment of objective function ΔW^{ν}. There are solved two problems of structural mechanics at each step. The first problem is determination of statically admissible residual moments $\boldsymbol{M}_r^{\nu}$ of plate at shakedown state (3.2) from load $\boldsymbol{F}_{sup}^{\nu}$, $\boldsymbol{F}_{inf}^{\nu}$ for each ν-th step. When $\boldsymbol{F}_{sup}^{\nu}$, $\boldsymbol{F}_{inf}^{\nu}$ are known, the solution of the problem (3.2) ν-th stage is $\boldsymbol{M}_r^{*\nu}$. For the optimal solution of the problem (3.2) plastic multipliers $\boldsymbol{\lambda}^{*\nu}$ are obtained, which are used for calculation of plastic strains $\boldsymbol{\Theta}_p^{*\nu}$ according to formula $\boldsymbol{\Theta}_{pkl}=2\sum_{j}\lambda_{kl,j}\boldsymbol{\Phi}\boldsymbol{M}_{kl,j}$. After plastic strains $\boldsymbol{\Theta}_p$ are calculated, the second problem of structural mechanics is solved, i.e. verification of plate stiffness conditions (3.3). For taking into account possible unloading phenomenon of cross-sections, stricter stiffness conditions (3.5) must be checked. In case these conditions are satisfied, the next ν-th step of the problem (3.1)–(3.3) is executed. When the conditions (3.5) are violated, the increment of objective function ΔW^{ν} is reduced and solution procedure of the problem (3.1)–(3.2) is repeated. Optimal solution $\boldsymbol{F}_{sup}^{*\nu}$, $\boldsymbol{F}_{inf}^{*\nu}$ is achieved when one of conditions (3.5), which are not violated, is satisfied as equality. According to proposed algorithm only rational solution is obtained, as solution results of the problem (3.1)–(3.2) depends on choice of initial point. Larger-scale flowchart of the problem (3.1)–(3.3), (3.5) solution algorithm is presented in Fig 2.

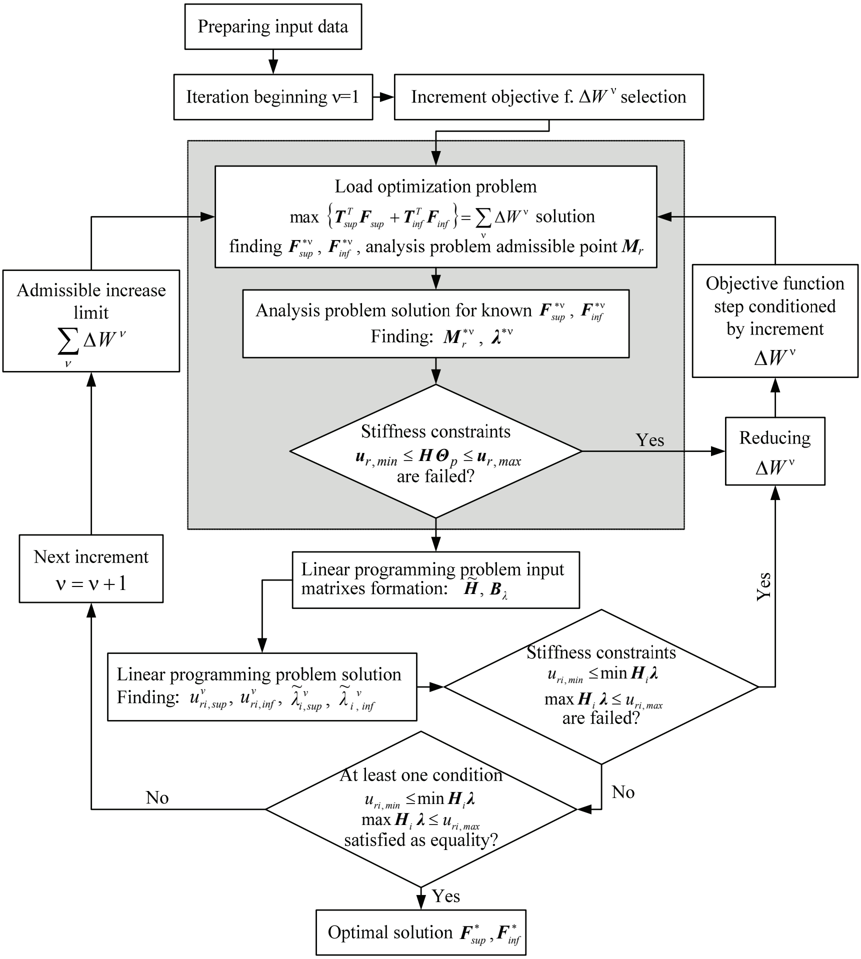


Fig 2. Flowchart of the load optimization problem (3.1)–(3.3), (3.5) solution algorithm

Solution algorithm of the second problem formulation is mainly the same.

Numerical experiments for circular plates were performed according to the first formulation of load optimization problem mathematical model (3.1)–(3.3), (3.5) (Fig 3). Equilibrium finite elements, which ensure better satisfaction of equilibrium equations, were used for plate discretization.

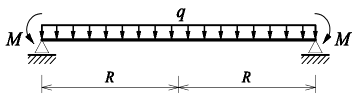
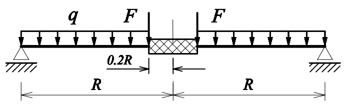
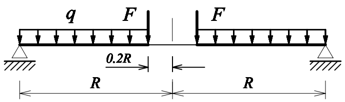
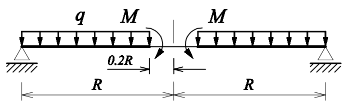
Plate design scheme	Initial data/ results
a) $0 \le q \le q_{sup}$, $0 \le M \le M_{sup}$ 	Limit deflection of the plate: $u_{ri,max} = 0.84\, M_0 R / \mathcal{K}$. Optimal solution: $q_{sup}^{*v} = 5.7716 M_0 R^{-2}$, $M_{sup}^{*v} = 0.8091\ M_0$.
b) $0 \le q \le q_{sup}$, $0 \le F \le F_{sup}$ 	Limit deflection of the plate: $u_{ri,max} = 3.4\, M_0 R / \mathcal{K}$. Optimal solution: $q_{sup}^{*v} = 6.7501 M_0 R^{-2}$, $F_{sup}^{*v} = 0.0456\ M_0$.
c) $0 \le q \le q_{sup}$, $0 \le F \le F_{sup}$ 	Limit deflection of the plate: $u_{ri,max} = 0.1 M_0 R / \mathcal{K}$. Optimal solution: $q_{sup}^{*v} = 5.7472 M_0 R^{-2}$, $F_{sup}^{*v} = 0.0290\ M_0$.
d) $0 \le q \le q_{sup}$, $0 \le M \le M_{sup}$ 	Limit deflection of the plate: $u_{ri,max} = 0.73 M_0 R / \mathcal{K}$. Optimal solution: $q_{sup}^{*v} = 5.7011 M_0 R^{-2}$, $M_{sup}^{*v} = 0.1481 M_0$.

Fig 3. Object of numerical examples for shakedown load optimization problem

Shortcoming of the proposed algorithm is that stiffness conditions (3.3) or (3.11) do not have direct influence to the solution of the problem (3.1)–(3.3). Conditions (3.3) or (3.11) perform verification function in respect to stiffness conditions. Thus, such algorithm would not serve for bar-systems, where often yield conditions are linear. For plates with Hubert-Mises yield conditions proposed algorithm worked stabile and obtained results were practically the same for calculations started from different initial points.

Search for creation of the more successful algorithm made to include plastic multipliers $\boldsymbol{\lambda}$ into the set of the main unknowns in mathematical models. That is done in the fourth chapter implementing solution of plate optimal parameter distribution problems.

4. Optimal Shakedown Design of Plates under Stiffness Constraints

In the fourth chapter, optimal distribution of limit bending moments or characteristic dimension of cross-section for adapted bending plate under strength and stiffness constraints is to be found.

The first formulation of the problem. Following mathematical model for determining optimal distribution of adapted plate parameters is created on the basis of general mathematical models of adapted structures under strength and stiffness constraints (Table 1):

find
$$\min \sum_k L_k M_{0k} = \min \boldsymbol{L}^T \boldsymbol{M}_0 \tag{4.1}$$

subject to
$$\left.\begin{aligned} &\min \frac{1}{2}\boldsymbol{M}_r^T \boldsymbol{D} \boldsymbol{M}_r \\ &\boldsymbol{A}\boldsymbol{M}_r = \boldsymbol{0}, \\ \varphi_{kl,j} = (M_{0k})^2 - &\left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right)^T \boldsymbol{\Phi} \left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right) \ge 0, \end{aligned}\right\} \tag{4.2}$$

$$M_{0k} \ge 0, \tag{4.3}$$

$$\boldsymbol{\Theta}_p = \left(\boldsymbol{\Theta}_{pkl}\right)^T, \ \boldsymbol{\Theta}_{pkl} = \sum_j \left[\nabla \varphi_{kl,j} \left(\boldsymbol{M}_{ekl,j} + \boldsymbol{M}_{rkl}\right)\right]^T \lambda_{kl,j},$$

$$\lambda_{kl,j} \ge 0, \ \boldsymbol{\lambda}_j = (\lambda_{kl,j})^T, \ k \in K, \ l \in L, \ j \in J; \tag{4.4}$$

$$\boldsymbol{u}_{r,min} \le \boldsymbol{H}\boldsymbol{\Theta}_p \le \boldsymbol{u}_{r,max}. \tag{4.5}$$

Components of the vector $\boldsymbol{L} = \left(L_1, L_2, ..., L_s\right)^T$ are areas of finite elements of the plate discrete model.

Unknowns of the problem (4.1)–(4.5) are vectors of limit $\boldsymbol{M}_0$, residual $\boldsymbol{M}_r$ bending moments, plastic multipliers $\boldsymbol{\lambda}_j$ ($j \in J$). Vectors $\boldsymbol{u}_{r,min}$, $\boldsymbol{u}_{r,max}$ are known in advance, they restrict variation of the residual displacements $\boldsymbol{u}_r = \boldsymbol{H}\boldsymbol{\Theta}_p$. Usually total deflections are restricted in the design codes, at that point stiffness conditions (4.5) would obtain following form:

$$\boldsymbol{u}_{min} \le \boldsymbol{u}_{e,inf} + \boldsymbol{u}_r, \quad \boldsymbol{u}_{e,sup} + \boldsymbol{u}_r \le \boldsymbol{u}_{max}. \tag{4.6}$$

Solving the problem (4.2) plastic multipliers $\boldsymbol{\lambda}$ are determined, though the fact of their determination is not defined directly. The relation between residual moments $\boldsymbol{M}_r$ and plastic strains $\boldsymbol{\Theta}_p$ (plastic multipliers $\boldsymbol{\lambda}$) is not used in mathematical model (4.1)–(4.5).

During shakedown process residual displacements $\boldsymbol{u}_r$ vary non-monotonically. Therefore condition (4.5) can be insufficient and following problem should be solved:

$$u_{ri,inf} = \min \boldsymbol{H}_i \tilde{\boldsymbol{\lambda}} \,, \quad u_{ri,sup} = \max \boldsymbol{H}_i \tilde{\boldsymbol{\lambda}} \,, \quad i = 1,2,...,m \,, \tag{4.7}$$

$$\boldsymbol{B}_\lambda \tilde{\boldsymbol{\lambda}} = \boldsymbol{B}_r \boldsymbol{M}_r^* \,, \quad \sum_j \tilde{\boldsymbol{\lambda}}_j^T \tilde{\boldsymbol{C}} \le \tilde{D}_{max} \,, \quad \tilde{\boldsymbol{\lambda}} \ge \boldsymbol{0} \,. \tag{4.8}$$

Then

$$\boldsymbol{u}_{r,min} \le \boldsymbol{u}_{r,inf} \,, \quad \boldsymbol{u}_{r,sup} \le \boldsymbol{u}_{r,max} \,. \tag{4.9}$$

Summarizing, it should be noted that mathematical model of the first optimization problem formulation (4.1)–(4.5) (together with (4.7)–(4.9)) has verification character basically. Stiffness conditions do not have direct influence to the solution of the optimization problem.

The second formulation of the optimization problem is stated assuming that the main unknowns are $\boldsymbol{M}_0$ and $\boldsymbol{\lambda}$ (Fig 4.). Applying non-linear von Mises yield condition, influence matrix of residual moments $\boldsymbol{G} = \boldsymbol{K}\boldsymbol{A}^T \left(\boldsymbol{A}\boldsymbol{K}\boldsymbol{A}^T\right)^{-1} \boldsymbol{A}\boldsymbol{K} - \boldsymbol{K}$ depends on $\boldsymbol{M}_r$ ($\boldsymbol{M}_r = \boldsymbol{G}\boldsymbol{\Theta}_p$, $\boldsymbol{\Theta}_p = \left(\boldsymbol{\Theta}_{pkl}\right)^T$, $\boldsymbol{\Theta}_{pkl} = 2\sum_j \boldsymbol{\lambda}_{kl,j} \boldsymbol{\Phi} \boldsymbol{M}_{kl.j} =$ $= \left[\nabla \varphi_{kl,j}\left(\boldsymbol{M}_{kl,j}\right)\right]^T \lambda_{kl,j}$). Then, mathematical model of the optimization problem reads:

find

$$\min \left\{ \sum_k L_k M_{0k} - \sum_k \sum_l \sum_j \lambda_{kl,j} \left[\left(M_{0k}\right)^2 - \left(\boldsymbol{M}_{ekl,j} + \overline{\boldsymbol{G}}_{kl}\boldsymbol{\lambda}\right)^T \boldsymbol{\Phi} \left(\boldsymbol{M}_{ekl,j} + \overline{\boldsymbol{G}}_{kl}\boldsymbol{\lambda}\right) \right] \right\} \tag{4.10}$$

subject to

$$\varphi_{kl,j} = \left(M_{0k}\right)^2 - \left(\boldsymbol{M}_{ekl,j} + \overline{\boldsymbol{G}}_{kl}\boldsymbol{\lambda}\right)^T \boldsymbol{\Phi} \left(\boldsymbol{M}_{ekl,j} + \overline{\boldsymbol{G}}_{kl}\boldsymbol{\lambda}\right) \ge 0 \,, \tag{4.11}$$

$$\lambda_{kl,j} \ge 0 \,, \ \boldsymbol{\lambda}_j = (\lambda_{kl,j})^T \,, \ \boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, ..., \boldsymbol{\lambda}_p\right)^T \,, \ k \in K \,, \ l \in L \,, \ j \in J \,; \tag{4.12}$$

$$M_{0k} \ge 0 \,, \tag{4.13}$$

$$\boldsymbol{u}_{r,min} \le \overline{\boldsymbol{H}}\boldsymbol{\lambda} \le \boldsymbol{u}_{r,max} \,. \tag{4.14}$$

Here matrix $\overline{\boldsymbol{G}}$ is obtained multiplying matrix $\boldsymbol{G}$ by gradient matrices of yield conditions $\left[\nabla \boldsymbol{\varphi}_j\right]$:

$$\overline{\boldsymbol{G}} = \left[\boldsymbol{G}\left[\nabla \boldsymbol{\varphi}_1\right], \boldsymbol{G}\left[\nabla \boldsymbol{\varphi}_2\right], ..., \boldsymbol{G}\left[\nabla \boldsymbol{\varphi}_p\right] \right]. \tag{4.15}$$

Matrix $\overline{\boldsymbol{H}}$ is determined analogously:

$$\overline{\boldsymbol{H}} = \left[\boldsymbol{H}\left[\nabla \boldsymbol{\varphi}_1\right], \boldsymbol{H}\left[\nabla \boldsymbol{\varphi}_2\right], ..., \boldsymbol{H}\left[\nabla \boldsymbol{\varphi}_p\right] \right]. \tag{4.16}$$

Applying Rosen algorithm for implementation of the mathematical model (4.10)–(4.14), matrices $\left[\nabla \boldsymbol{\varphi}_1\left(\overline{\boldsymbol{M}}_{e1}+\overline{\boldsymbol{M}}_r\right)\right]$, $\left[\nabla \boldsymbol{\varphi}_2\left(\overline{\boldsymbol{M}}_{e2}+\overline{\boldsymbol{M}}_r\right)\right], \ldots, \left[\nabla \boldsymbol{\varphi}_p\left(\overline{\boldsymbol{M}}_{ep}+\overline{\boldsymbol{M}}_r\right)\right]$, which are included into expressions (4.15), (4.16), are calculated using argument $\left(\overline{\boldsymbol{M}}_{ej}+\overline{\boldsymbol{M}}_r\right)$ obtained in one solution step τ behind.

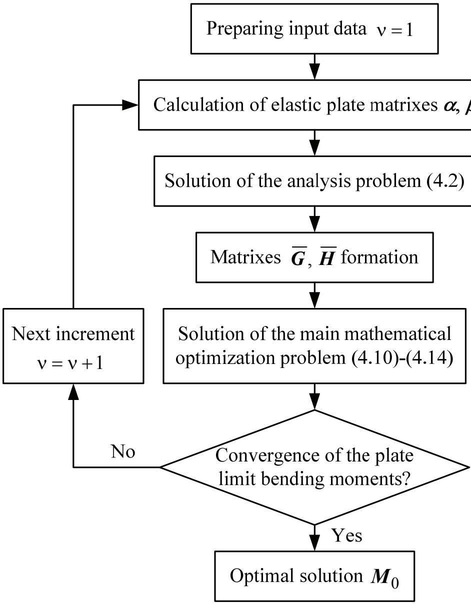


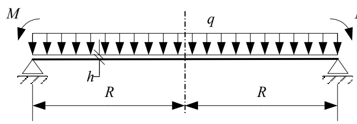
Fig 4. Solution algorithm of the plate optimization problem

About efficiency of optimality criterion. In the case of plate minimal volume, volume function $\boldsymbol{L}^T\boldsymbol{h}$ is introduced into optimality criterion directly. The main problem (4.10)–(4.14), containing equilibrium, geometrical and physical (yield) conditions in its constraints, allows to proceed to criterion of plate minimal volume directly (for plates with solid or sandwich cross section).

Numerical experiments were performed for circular plate with solid cross-section according to both formulations of optimal design problem (Fig 5).

Convergence of the plate elements limit bending moments

STAGE	$M_{0,1}$	$M_{0,2}$	$M_{0,3}$	$M_{0,4}$	$M_{0,5}$	$M_{0,6}$	$\boldsymbol{L^T M_0}$
$v = 1$	53.06	52.77	52.53	52.29	52.05	51.81	132.78
$v = 2$	52.92	52.18	51.40	49.80	48.85	47.90	125.93
$v = 3$	52.91	52.10	51.35	49.74	48.43	47.39	125.21
$v = 4$	52.89	51.91	51.23	49.61	47.39	46.11	123.40
$v = 5$	52.77	51.71	51.07	49.39	47.08	44.23	121.52
$v = 6$	52.37	51.56	50.66	48.72	46.02	42.64	119.07



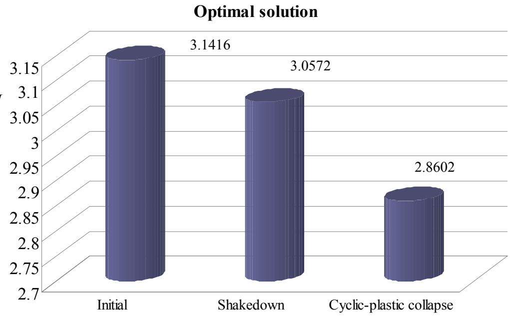


Fig 5. Object of numerical examples for optimal shakedown design problem

5. The Main Results and Conclusions

1. The features of recent shakedown theory development and achievements in field of structural optimization show topicality and need of research on structural optimization.
2. In structural engineering real effects for structure are cyclic and often application of shakedown theory is necessary requirement in evaluation of structural safety.
3. It is proved that strain compatibility equations of deformable body mechanics are Kuhn-Tucker conditions of mathematical programming.
4. It is shown that Rosen optimality criterion and Kuhn-Tucker conditions are identical. Solving non-linear mathematical programming problem, that allows to obtain also solution of its dual problem at once.
5. General mathematical models of structure shakedown optimization problems with non-linear yield conditions are improved.
6. In the case of Hubert-Mises yield conditions, technique for construction of plate elastic force locus by means of equilibrium finite elements is created.
7. It is shown that complementary slackness conditions do not allow to evaluate local unloading phenomenon of cross-sections during shakedown process.
8. Improved technique for deflection variation bound calculation of bending plates is proposed.

9. Mathematical models of adapted plate verification and optimal design problems with Hubert-Mises yield conditions are created. It is shown that shakedown optimization problem is not classical mathematical programming one.
10. Possibilities of introducing plate shakedown analysis problems into the mathematical models (design and verification aspect) are found.
11. Step-by-step solution algorithm for plate shakedown load and optimal parameter distribution problems with strength and stiffness constraints is created.
12. Software for non-linear optimization problem solution of bending plates, which are solved in the dissertation, is created with possible integration to the other software.

A List of Publications on the Subject of the Dissertation

1. ATKOČIŪNAS J.; JARMOLAJEVA E.; MERKEVIČIŪTĖ D. Optimal shakedown loading for circular plates. *Structural and Multidisciplinary Optimization*, Vol. 27, No 3. Berlin: Springer–Verlag, 2004. p. 178–188. ISSN 1615–147X. (ISI Master List).
2. ATKOČIŪNAS J.; RIMKUS L.; JARMOLAJEVA E. Shakedown loading optimization for circular plates. Metal Structures – Design, Fabrication, Economy. In: *Proceedings of the International Conference on Metal Structures–ICMS–03*. Miscolc, Hungary, April 3–5, 2003. Rotterdam: Millpress, Netherlands. p. 251–258. ISBN 90 77017 75 5. (ISI Proceedings).
3. CHRAPTOVIČ (JARMOLAJEVA) E.; ATKOČIŪNAS J. Role of Kuhn-Tucker conditions in elasticity equations in terms of stresses. *Statyba* (Civil Engineering), Vol. VI, No 2. Vilnius: Technika, 2000. p. 104–112 (in Russian). ISSN 1392–1525.
4. CHRAPTOVIČ E.; ATKOČIŪNAS J. Mathematical programming applications pecularities in shakedown problem. *Statyba* (Civil Engineering), Vol. VII, No 2. Vilnius: Technika, 2001. p. 106–114 (in Russian). ISSN 1392–1525.
5. JARMOLAJEVA E.; ATKOČIŪNAS J. Shakedown loading optimization under constrained residual displacements – formulation and solution for circular plates. *Journal of Civil Engineering and Management*, Vol. VIII, No 1. Vilnius: Technika, 2002. p. 54–67. ISSN 1392–3730.
6. NAGEVIČIUS J.; RIMKUS L.; CHRAPTOVIČ E. Energy interpretation of complementarity condition at shakedown. In: *Proceedings of 6th International Conference on Modern Building Materials, Structures and Techniques*, May 19–22, 1999, Vilnius, Lithuania. Vilnius Gediminas technical university, Vilnius: Technika, 1999. p. 217–222 (in Lithuanian). ISBN 9986–05–378–1.
7. JARMOLAJEVA E.; SKARŽAUSKAS V.; ATKOČIŪNAS J. Analysis of optimality criteria and gradient projection method for optimal shakedown design. In: *Proceedings of International Conference "Mechanics–2001"*. Kaunas: Technologija, 2001. p. 166–171 (in Lithuanian). ISBN 9986–13–955–4.

8. CHRAPTOVIČ E.; ATKOČIŪNAS J. Shakedown of structures: mechanical interpretation of Rosen optimality criteria. In: *Proceedings of 7th International Conference on Modern Building Materials, Structures and Techniques*, May 16–18, 2001, Vilnius, Lithuania. CD–ROM (in Lithuanian). ISBN 9986–05–440–0.
9. JARMOLAJEVA E. Optimal shakedown design of axis-symmetric plates. In: *Proceedings of the 4th Conference of Lithuanian Young Scientists "Lithuania without science – Lithuania without future"*, March 30, 2001, Vilnius, Lithuania. Vilnius: Technika, 2001. p. 55–66 (in Lithuanian).
10. E. CHRAPTOVIČ. Mathematical model for estimating of residual displacements at shakedown. In: *Proceedings of the 3th Conference of Lithuanian Young Scientists "Lithuania without science – Lithuania without future"*, March 31, 2000, Vilnius, Lithuania. Vilnius: Technika, 2000. p. 9–15 (in Lithuanian).
11. SKARŽAUSKAS V.; ATKOČIŪNAS J.; CHRAPTOVIČ E. Load optimization of elastic-plastic frames in structural mechanics. Development and investigation of steel and wood structures – collection of scientific works: Kazan: KGSA. 1999. p. 56–62 (in Russian).

Briefly about the author

Ela Jarmolajeva was born in Vilnius region on June 3, 1972. BSc and MSc in Civil Engineering, Faculty of Civil Engineering, Vilnius Gediminas Technical University in 1995 and 1997, respectively. Since 1998 she began PhD studies at Vilnius Gediminas Technical University in a field of Technological Sciences in a branch of Civil Engineering. In 2004–2006 she worked as young science employee at the Science Institute of Human Safety and assistant at the Department of Labour Safety and Fire Protection of Vilnius Gediminas Technical University. Since 2006 she works scientific and pedagogical work at the Structural Mechanics Department of Vilnius Gediminas Technical University.

LENKIAMŲ PLOKŠČIŲ OPTIMIZACIJA PRISITAIKOMUMO SĄLYGOMIS

Mokslo problemos aktualumas. Matematika ir mechanika visada turėjo įtakos viena kitos raidai, keliant naujas mokslines idėjas ir jas įgyvendinant. Vienas tokio glaudaus ryšio pavyzdžių yra matematinio programavimo teorijos platus taikymas ekstreminiams mechanikos uždaviniams spręsti. Ši teorija įgalina formuluoti ir prisitaikančių konstrukcijų optimizavimo uždavinius bei juos realizuoti. Disertaciniame darbe, pasitelkus matematinio programavimo teoriją, išplėtota ne tik nauja lenkiamų prisitaikančių plokščių optimizavimo metodika, bet atskleistos Kuno ir Takerio sąlygų sąsajos su deformuojamo kūno mechanikos deformacijų darnos (Sen-Venano) lygtimis bei asociatyvinio tekėjimo dėsnio priklausomybėmis. Konstrukcijų skaičiavimas įvertinant plastines deformacijas, kartu atsižvelgiant į apkrovų kintamą kartotinį poveikį, leidžia susieti statybinės mechanikos optimizavimo uždavinius su realiu projektavimo procesu. Prisitaikiusi ideali tampri plastinė konstrukcija tenkina stiprumo sąlygas ir yra saugi ciklinio plastinio suirimo atžvilgiu. Tačiau ji gali netenkinti eksploatacinių reikalavimų. Todėl lenkiamos plokštės parametrų arba apkrovos kitimo ribų optimizavimo uždavinių diskretiniuose matematiniuose modeliuose turi būti ne tik stiprumo, bet ir standumo sąlygos apribojimai. Netiesinių uždavinių matematiniai modeliai, sudaryti taikant pusiausvirų baigtinių elementų metodą, sprendžiami iteraciniu būdu, pasitelkus Rozeno projektuojamųjų gradientų algoritmą. Darbui būdinga tai, kad matematinio programavimo teorija optimizavimo problemos nagrinėjimą lydi nuo matematinio modelio sudarymo iki jo skaitinio išsprendimo, kartu atskleidžiant taikomo Rozeno algoritmo optimalumo kriterijaus mechaninę prasmę. Tuomet ir iškilojo netiesinio programavimo atveju iš karto (analogiškai tiesinio programavimo atvejui) nustatomos dualiųjų kintamųjų reikšmės. Tai spartina prisitaikančių konstrukcijų optimizavimo uždavinių realizaciją, ypač turint galvoje tai, kad disipatyvinių sistemų įtempių ir deformacijų būvis priklauso nuo apkrovimo istorijos.

Pagrindinis darbo tikslas. Plėtoti matematinio programavimo teorijos taikymą kieto deformuojamo kūno mechanikoje: parodyti, jog ekstremumo uždaviniui, suformuluotam pagal Kastiljano principą, Kuno ir Takerio sąlygos yra deformacijų darnos lygtys. Sukurti naujus prisitaikančių tampriųjų plastinių lenkiamų plokščių parametrų ir apkrovos pasiskirstymų optimizavimo, ribojant jos įlinkius, metodus ir efektyvius uždavinių sprendimo algoritmus.

Tyrimų objektas. Kuno ir Takerio sąlygos tamprumo ir plastiškumo teorijos ekstremumo uždaviniuose. Idealios tamprios plastiškos izotropinės plokštės optimizavimo uždavinių (projektinis ir patikrinamasis aspektai) matematiniai modeliai ir sprendimo algoritmai.

Tyrimo metodai. Pasitelkiami ekstreminiai energiniai deformuojamo kūno mechanikos principai ir matematinio programavimo teorija. Optimizavimo uždavinių diskretiniai matematiniai modeliai sudaromi pusiausvirųjų baigtinių elementų metodu. Tyrimai atlikti ir skaitinių eksperimentų rezultatai gauti, laikantis mažų poslinkių prielaidos.

Darbo uždaviniai: 1) atskleisti Kuno ir Takerio sąlygų ir prisitaikančių sistemų pagrindinių lygčių sąsajas; 2) atlikti prisitaikančių konstrukcijų, jų tarpe lenkiamų plokščių, šiuolaikinių skaičiavimo metodų analizę; 3) sudaryti plokščių analizės ir optimizavimo uždavinių bendruosius matematinius modelius; 4) nustatyti lenkiamų plokščių liekamųjų įlinkių kitimo ypatumus ir sudaryti jų ribų skaičiavimo uždavinio matematinį modelį; 5) atlikti lenkiamų plokščių apkrovos kitimo ribų optimizavimą prisitaikomumo sąlygomis. 6) optimizuoti lenkiamos plokštės parametrų pasiskirstymą, įvertinant stiprumo ir standumo sąlygas apribojimus; 7) sukurti optimizavimo uždavinių, esant netiesinėms Hubero-Mizeso takumo sąlygoms, sprendimo algoritmus.

Mokslinis naujumas. 1) Darbe atskleidžiamos matematinio programavimo teorijos, plačiai paplitusios kaip ekstreminių uždavinių sprendimo metodas, naujos galimybės formuluojant prisitaikomumo teorijos uždavinius ir juos skaitiškai sprendžiant. 2) Išaiškintas Kuno ir Takerio sąlygų vaidmuo formuluojant bendrąsias deformuojamo kūno mechanikos lygtis įtempiais (tamprių plastinių konstrukcijų deformacijų darnos lygtys). Parodyta, kad ribinės pusiausvyros uždaviniuose Kuno ir Takerio sąlygos apima savyje ir asociatyvinio tekėjimo dėsnio priklausomybes. 3) Rozeno projektuojamųjų gradientų metodo optimalumo kriterijaus mechaninės prasmės išaiškinimas įgalina iškarto gauti dualaus konstrukcijos netiesinio analizės uždavinio sprendinį. 4) Sudarytas pagerintas tamprios plastinės lenkiamos plokštės liekamųjų įlinkių kitimo ribų skaičiavimo uždavinio matematinis modelis įgalina tiksliau formuoti standumo sąlygas prisitaikančių plokščių optimizavimo uždaviniuose. Tuo pačiu sumažinta neleidžiančių įvertinti fizinio nusikrovimo reiškinio matematinio programavimo griežtumo sąlygų įtaka. 5) Sudaryti nauji lenkiamų plokščių optimizavimo (projektinis ir patikrinamasis aspektai) prisitaikomumo sąlygomis uždavinių matematiniai modeliai.

Darbo aprobavimas ir publikacijos. Pagrindiniai disertacinio darbo rezultatai aptarti devyniose mokslinėse konferencijose, paskelbti vienuolikoje mokslo straipsnių. Tarp jų vienas leidinyje, įrašytame į Mokslinės informacijos instituto pagrindinį sąrašą (ISI Master List), vienas tarptautinių mokslo organizacijų organizuotų konferencijų pranešimų medžiagoje (ISI Proceedings), trys Lietuvos recenzuojamuose periodiniuose mokslo leidiniuose, įtrauktuose į bazes, nurodytas LMT 2006 m. aprobuotame sąraše.

Disertacijos apimtis. Lietuvių kalba parašytąją disertaciją sudaro: įvadas, keturi atskiri skyriai, išvados, literatūros (182 pozicijų) ir disertacijos autorės publikacijų (11 pozicijų) sąrašai. Disertacijos tekstas pateiktas 103 puslapiuose, tekste yra 14 paveikslų, 12 lentelių.

Disertacijos sandara. Prisitaikančių konstrukcijų bendriesiems principams skirtas 1-asis skyrius. Jame išnagrinėta prisitaikomumo teorijos raida, apžvelgtos šiuolaikinės jos vystymosi kryptys, aptarti bendrieji energiniai principai bei pateikti konstrukcijų ciklinio plastinio suirimo ir prisitaikymo būvio uždavinių matematiniai modeliai. Antrajame skyriuje išsamiau nagrinėjamos matematinio programavimo teorijoje gaunamų priklausomybių sąsajos su tamprumo ir plastiškumo teorijos pagrindinėmis lygtimis. Parodyta, kaip taikant Kuno ir Takerio sąlygas gaunamos tamprumo teorijos lygtys įtempiais bei asociatyvinio tekėjimo dėsnio išraiškos plastiškumo teorijoje. Trečiajame skyriuje nagrinėjamos prisitaikiusių plokščių apkrovos optimizavimo uždavinių formuluotės ir sprendimo algoritmai. Ketvirtas skyrius skirtas prisitaikiusių plokščių parametrų optimalaus pasiskirstymo uždavinių ir jų sprendimo algoritmų formulavimui. Darbe prisitaikiusios plokštės optimizavimo uždaviniai formuluojami taikant netiesinį matematinį programavimą. Penktame skyriuje pateiktos darbą apibendrinančios išvados.

Pagrindiniai darbo rezultatai ir išvados

1. Šiuolaikinės prisitaikomumo teorijos plėtros bruožai ir įdirbis teigia prisitaikančių konstrukcijų optimizavimo darbų aktualumą ir poreikį.
2. Statybos inžinerijoje realūs konstrukcijos poveikiai cikliški, todėl prisitaikomumo teorijos taikymas konstrukcijų saugumo įvertinime yra svarbus, dažnai būtinas reikalavimas.
3. Įrodyta, kad deformuojamo kūno mechanikos deformacijų darnos lygtys yra matematinio programavimo Kuno ir Takerio sąlygos.
4. Parodyta, kad Rozeno optimalumo kriterijus ir Kuno ir Takerio sąlygos yra tapatingi. Tai leidžia, sprendžiant netiesinio matematinio programavimo uždavinį, iš karto gauti ir jam dualaus uždavinio sprendinį.
5. Patobulinti prisitaikančių konstrukcijų optimizavimo uždavinių su netiesinėmis takumo sąlygomis bendrieji matematiniai modeliai.
6. Sukurta tamprių plokštės įrąžų hodografo sudarymo pusiausvirais baigtiniais elementais metodika, esant netiesinei Hubero-Mizeso takumo sąlygai.
7. Parodyta, kad matematinio programavimo griežtumo sąlygos neleidžia įvertinti vietinio skerspjūvių nusikrovimo prisitaikymo proceso metu.
8. Pasiūlyta patobulinta lenkiamų plokščių įlinkių ribų skaičiavimo metodika.
9. Sudaryti prisitaikančių plokščių, esant netiesinėms Hubero-Mizeso takumo sąlygoms patikrinamojo ir projektinio uždavinių matematiniai modeliai. Parodyta, kad prisitaikomumo būvio optimizavimo uždavinys nėra klasikinė matematinio programavimo problema.
10. Surastos plokščių prisitaikomumo būvio analizės uždavinių įjungimo į optimizavimo matematinius modelius (patikrinamasis ir projektinis aspektai) galimybės.
11. Sukurtas plokščių apkrovos ir parametrų optimalaus pasiskirstymo prisitaikomumo sąlygomis uždavinių, esant stiprumo ir standumo apribojimams, etapinis sprendimo algoritmas.
12. Sukurta lenkiamų plokščių netiesinių optimizavimo uždavinių, spręstų disertaciniame darbe, realizavimo kompiuterinė programa su integracijos galimybe su kitomis programomis.

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OPTIMIZATION OF BENDING PLATES AT SHAKEDOWN

Summary of Doctoral Dissertation
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