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Scientific Supervisor:
Prof Dr Habil Leonas SAULIS (Vilnius Gediminas Technical University, physical sciences, mathematics - 01P);
The dissertation is defended in the Council of Mathematical Sciences of Vilnius Gediminas Technical University:
Chairman:
Prof Dr Habil Raimondas ČIEGIS (Vilnius Gediminas Technical University, Physical Sciences, Mathematics - 01P).
Members:
Prof Dr Habil Romanas JANUŠKEVIČIUS (Vilnius Pedagogical University, Physical Sciences, Mathematics - 01P);
Assoc Prof Dr Aleksandras KRYLOVAS (Vilnius Gediminas Technical University, Physical Sciences, Mathematics - 01P);
Prof Dr Habil Antanas LAURINČIKAS (Vilnius University, Physical Sciences, Mathematics - 01P);
Prof Dr Habil Rimantas RUDZKIS (Institute of Mathematics and Informatics, Physical Sciences, Mathematics - 01P);
Opponents:
Prof Dr Habil Mindaugas BLOZNELIS (Vilnius University Physical Sciences, Mathematics - 01P);
Prof Dr Habil Jonas Kazys SUNKLODAS (Institute of Mathematics and Informatics, Physical Sciences, Mathematics - 01P);

The dissertation will be defended in the public meeting of the Council of Mathematical Sciences in the Senate Hall of Vilnius Gediminas Technical University at 2 p.m. on December 03, 2004 m. Address: Saulėtekio al. 11, LT - 10223, Vilnius - 40, Lithuania
Tel. (+370 5)274 49 52; (+370 5)274 49 56, fax (+370 5)270 01 12 The dissertation and the summary of doctoral dissertation was distributed on November 02, 2004. The dissertation is available at Vilnius Gediminas Technical University Library (Saulėtekio al. 14, Vilnius).

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Disertacija ginama eksternu.

Mokslinis vadovas
prof. habil. dr. Leonas SAULIS (Vilniaus Gedimino
Technikos universitetas, fiziniai mokslai, matematika - 01P);

Disertacija ginama Vilniaus Gedimino technikos universiteto ir Matematikos:
Pirmininkas:
prof. habil. dr. Raimondas ČIEGIS (Vilniaus Gedimino technikos
universitetas, fiziniai mokslai, matematika - 01P);
nariai:
1. prof. habil. dr. Rimantas RUDZKIS (Matematikos ir
informatikos institutas, fiziniai mokslai, matematika - 01P);
2. doc. dr. Aleksandras KRYLOVAS (Vilniaus Gedimino technikos
universitetas, fiziniai mokslai, matematika - 01P);
3. prof. habil. dr. Antanas LAURINČIKAS (Vilniaus Universitetas,
fiziniai mokslai, matematika - 01P);
4. prof. habil. dr. Romanas JANUŠKEVIČIUS (Vilniaus pedagoginis
universitetas, fiziniai mokslai, matematika - 01P);

Oponentai:
1. prof. habil. dr. Jonas Kazys SUNKLODAS (Matematikos
ir informatikos institutas, fiziniai mokslai, matematika - 01P);
2. prof. habil. dr. Mindaugas BLOZNELIS (Vilniaus universitetas
fiziniai mokslai, matematika - 01P);

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Adresas:Saulėtekio al. 11, 2040, Vilnius, Lietuva
Tel. 8 5 274 49 52; 8 5 274 49 56, faksas 8 5 270 01 12

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INTRODUCTION

Among the problems of limit theorems for large deviations in the probability theory and mathematical statistics a great many of works are spare to the asymptotic of distributions of the sums $S_n = X_1 + X_2 + \ldots + X_n$ of independent and dependent random variables (r.v.'s). The condition on the existence of exponential moments takes an extremely significant place here: there exist quantities $a > 0$ ir $\gamma \geq 0$ such that

$$\mathbf{E} \exp\left\{a|X_j|^{\frac{1}{1+\gamma}}\right\} < \infty, \qquad j = 1, 2, \ldots, n.$$

In case $\gamma = 0$, it is assumed that a random variable X_j satisfies the Cramer condition. In this case, the characteristic function (ch.f.) of the random variable X_j

$$f_{X_j}(t) = \mathbf{E} \exp\{itX_j\} \tag{1}$$

is analytical in the vicinity of the point $t = 0$. If random variable X_j, $j = 1, 2, \ldots, n$ satisfy the mentioned condition with $\gamma > 0$, it is said that they satisfy the Linnik condition. In this case, there exist moments of all orders of the r.v. X_j, however, their growth does not ensure analyticity of characteristic function $f_{X_j}(t)$ in the vicinity of zero point.

Without loss of generality, we will further assume that the mean of a r.v. S_n, $\mathbf{E}S_n = 0$ and variance $B_n^2 = \mathbf{D}S_n$. In large deviation theorems, asymptotic behaviour (convergence to a unit), the rate of convergence, and asymptotic expansion of the relation

$$D_n(x) := \frac{\mathbf{P}(S_n \geq B_n x)}{(1 - \Phi(x)) \exp\{\lambda_n(x)\}} \tag{2}$$

are considered as $x = \Lambda(n) \to \infty$, $n \to \infty$. Here $\lambda_n(x)$ is a converging Cramer - Petrov series (in case $\gamma = 0$) with the coefficients expressed through cumulants of a r.v. $Z_n = S_n/B_n$, where the the k^{th} - order cumulant $\Gamma_k(X)$ of the r.v. X is defined by the equality

$$\Gamma_k(X) = \frac{1}{i^k} \frac{d^k}{dt^k} \ln f_X(t)\bigg|_{t=0}, \qquad k = 1, 2, \ldots, \tag{3}$$

and $\lambda_n(x)$ is a polynomial with the coefficients coincident with Cramer - Petrov coefficients, and the number of terms dependent on the variable γ included in the abovementioned condition. Besides, $\Phi(x)$ is the standard normal distribution $N(0,1)$.

In the analysis of the relation $D_n(x)$ defined by equality (2), H.Cramer's work (1938) takes an important place. As the summands of the sums $S_n = \sum_{j=1}^{n} X_j$ are independent identically distributed random variables, V.V.Petrov (1972) obtained the optimal result presented in the monograph "Sums of Independent Random Variables". Note that in this monograph large deviation theorems valid in the Cramer zone are presented, i.e., when the random variables X_j, $j = 1, 2, \ldots, n$ satisfy Cramer condition. the asymptotic analysis of large deviation relation $D_n(X)$, defined by equality(2), is much more complicated in case individual summands X_j, $j = 1, 2, \ldots, n$ of the sum S_n are independent identically distributed and satisfy the Linnik condition $\mathbf{E}\exp\left\{a|X_j|^{\frac{1}{1+\gamma}}\right\} < \infty$, $\gamma > 0$. Asymptotic (convergence to a unit) and the rate of convergence of the relation $D_n(x)$ in the power Linnik zones were completely investigated by I.A. Ibragimov, Ju.V. Linnik in the monograph "Independent and Stationary Sequences of Random Variables", (1965) and by S.V.Nagajev in the survey paper (1975). In these and other works large deviation theorems have been obtained by rather complicated analytical saddle point method, and as a rule, for sums of independent identically distributed random variables. This is the simplest case that allows one to conceive the general view of large deviation probabilities.

The next step in the problems of large deviation theorems was made in L.Saulis works (1969, 1973) in which asymptotic expansions of relation $D_n(x)$, defined by equality (2), were obtained in case individual summands of the sum S_n of independent identically distributed random variables satisfy the above Cramer and Linnik conditions. The works of L.Saulis, E.Misevicius (1973), L.Saulis and A.Nakas (1973) and others were designed to these issues.

V.Statulevicius (1966) proposed the method of cumulants (semiinvariants) to consider large deviation probabilities of various statistics under requirement that any random variable X with the mean $\mathbf{E}X = 0$ and variance $\mathbf{D}X = 1$ satisfy the condition: the exist quantities $\gamma \geq 0$ ir $\Delta > 0$, such that

$$|\Gamma_k(X)| \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad k = 3, 4, \ldots, \qquad (\mathbf{S}_\gamma)$$

where $\Gamma_k(X)$ is the k^{th} - order cumulant, defined by equality (3), of random variable X. Note that in this work a lemma proved in the case where $\gamma = 0$, i.e., the function $\varphi_X(z) = \mathbf{E}\exp\{zX\}$ generating moments of the random variable X in an analytical one in the domain $|z| < \Delta$. In 1978m. V.Statulevicius and his disciples R.Rudzkis and L.Saulis proved the *general lemma of large devia-*

tions, i.e, in case the random variable X satisfies the Statulevicius condition $(\mathbf{S}_\gamma)$ (V.Statulevicius, R.Rudzkis, L.Saulis (1978)). This condition can be easily verified for various multilinear forms, therefore it is very convemient for asymptotic analysis of large deviations of various statistics. Frequently, instead of accurate equalities of large deviations, less accurate exponential inequalities were used. These were proved by R.Bentkus, R.Rudzkis (1980), under a requirement that any random variable X should satisfy the abovementioned Statulevicius condition $(\mathbf{S}_\gamma)$. Both the general lemma of large deviations and exponential inequalities define the gist of the cumulant method. Application of these lemmas to prove large deviation theorems of various statistics is presented in the monograph "Limit Theorems for Large Deviations", by L.Saulis and V.Statulevicius (1989). It has been shown in it that, in order to get large deviation theorems or exponential inequalities for a statistic considered, it is necessary to find the estimates of cumulants of this statistic.

The method of cumulants rendered an apportunity to obtain large deviation theorems for sums of independent non-identically distributed random variables and dependent random variables in the Cramer and power Linnik zones, as well as for multilinear forms, multiple stochastic integrals, for the estimates of spectral density of sequences, for U-statistics, etc.

A.Žemaitis (1974) obtained the asymptotic expansion for the probability $\mathbf{P}(X \geq x)$ of the random variable X in the Cramer zone of large deviations, in case this variable fulfils the condition $(\mathbf{S}_\gamma)$ with $\gamma = 0$. Based on this result, the asymptotic expansion for the distribution of the sum of independent identically distributed random variables has been obtained, and this result was coincident with that of L.Saulis' (1969) in a separate case.

In 1997 L.Saulis proved the general lemme of asymptotic expansion in the Cramer and power Linnik zones, in case the random variable X satisfies the condition $(\mathbf{S}_\gamma)$, $\gamma \geq 0$, and he also presented the structure of the remainder term.

Main task. This work is anued at obtaining asymptotic expansions for the distribution and its density functions of the normed random variables

$$Z_n = S_n/B_n, \qquad S_n = \sum_{j=1}^{n} \xi_j^{(n)}, \qquad B_n^2 = \sum_{j=1}^{n} {\sigma_j^{(n)}}^2, \tag{4}$$

in the Cramer and power Linnik zones, where individual summands $\xi_j^{(n)}$, $j = 1, 2, \dots, n$, with the mean $\mathbf{E}\xi_j^{(n)} = 0$ and variances ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2 > 0$, satisfy the generalized S.N.Bernstein condition $(\mathbf{B}_\gamma)$: there exist quantities $\gamma \geq 0$ and

$K_j^{(n)}$ such that

$$\left|\mathbf{E}\xi_j^{(n)^k}\right| \leq (k!)^{1+\gamma}(K_j^{(n)})^{k-2}\sigma_j^{(n)^2}, \qquad k = 3, 4, \ldots, \qquad (\mathbf{B}_\gamma)$$

as well as at finding the estimates of the remainder terms of asymptotic expansions.To realize these aims, the methods of cumulants and characteristic functions have been used. It is rather a complicated problem that was solved for the first time in the problems of limit theorems for large deviations in the probability theory and mathematical statistics.

Novelty and the practical use of the work. The novelty and originality of the work consists in the fact that in order to obtain asymptotic expansions with optimal values of the remainder terms in the zone of large deviations, along with the cumulant method the classical method of characteristic functions has to be used. In addition, when solving the problems stated in the work, other than the well known results in the problems of limit theorems of the probability theory and mathematical statistics, we have to estimate constants. Technically it is frequently rather a complicated task.

The results obtained in the work have good opportunities to be applied in probability theory, mathematical statistics, econometric, etc. That is illustrated in the last section of the work in which theorems of large deviations are proved in the summation of weighted random variables with weights as well as discounted limit theorems.

Structure of dissertation. The dissertation consists of introduction, five chapters, conclusions (88 pp.), and 75 titles of references. Dissertation is written in lithuanian.

CONTENTS

The first chapter of the dissertation **"Survey of researches "** gives a wide description of researches on the topic of dissertation, performed in Lithuania and abroad, and an exhaustive analysis of literature is made. The works are indicated, on the basis of which the problems posed in the dissertation have been solved and new proposition obtained.

In the second chapter **General lemmas** general lemmas have been formulated as well as, an exponential inequality for any r.v. ξ with mean $\mathbf{E}\xi = 0$, unit variance $\mathbf{D}\xi = 1$, and distribution function $F_\xi(x) = \mathbf{P}(\xi < x)$, as this r.v. satisfies the condition $(\mathbf{S}_\gamma)$. Based on these general lemmas, in case a cumulant of any r.v. ξ is defined by equality (3), the major results of the dissertation have been obtained. That is, we obtain asymptotic expansions of the distribution and its density functions of normed centered r.v. Z_n, defined by equality (4), in the Cramer and power Linnik zones for large deviations with the estimates of remainder terms.

The third chapter **"Asymptotic expansions for the distribution function of the sum of random variables in the series scheme in the large deviation Cramer and power Linnik zones "** illustrates the asymptotic expansion obtained for the distribution function of the sum of random variables in the series scheme. Theorems on estimates of the reminder term of the asymptotic in the Cramer zone, as $\gamma = 0$, and the power Linnik zone, as $\gamma > 0$ of distributions of the r.v. Z_n with the summands $\xi_j^{(n)}$ satisfying condition (B_γ) have been proved. The exponential inequality of probability $\mathbf{P}\{\pm Z_n \geq x\}$ of the r.v. Z_n, defined by equality (5), has been obtained as well.

In the fourth chapter **"Asymptotic expansions of the distribution density function for the sum of random variables in the series scheme in large deviation zones"** the asymptotic expansion for the distribution density function of the r.v. Z_n as well as the estimates of the remainder terms of this expansion have been found and theorems proved.

In the fifth chapter **"Applications of general theorems"** theorems have been obtained and proved which were applied to particular case, i.e., when random variables are summed with weights, as well as, discounted limit theorems.

Obtained results. The results of dissertation have been obtained in the third, fourth, and fifth chapters.

"Asymptotic expansions for the distribution function of the sum of random variables in the series scheme in the large deviation Cramer and power Linnik zones ". Let $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$ be a sequence of random variables with means $\mathbf{E}\xi_j^{(n)} = 0$, and variances ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2 > 0$, $j = \overline{1, n}$.

Denote:

$$S_n = \sum_{j=1}^{n} \xi_j^{(n)}, \qquad B_n^2 = \sum_{j=1}^{n} {\sigma_j^{(n)}}^2, \qquad Z_n = \frac{S_n}{B_n}, \tag{5}$$

$$F_{Z_n}(x) = \mathbf{P}(Z_n < x), \qquad p_{Z_n}(x) = \frac{d}{dx} F_{Z_n}(x). \tag{6}$$

Next, let $f_\xi(t)$ and $\Gamma_k(\xi)$ denote the characteristic function (ch.f.) of a r.v. ξ and the k^{th} - order cumulant that are defined by equalities (1) and (3), respectively.

It is required that r.v.'s $\xi_j^{(n)}$ satisfy the generalized Bernstein condition $(\mathbf{B}_\gamma)$. If this condition is fulfilled, then the inequality

$$|\Gamma_k(\xi_j^{(n)})| \leq (k!)^{1+\gamma} (2(K_j^{(n)} \vee \sigma_j^{(n)}))^{k-2} {\sigma_j^{(n)}}^2, \qquad k = 3, 4, \ldots, \tag{7}$$

holds for the k^{th} - order cumulant $\Gamma_k(\xi_j^{(n)})$ of r.v.s $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$.

The proof of this inequality has been obtained and presented in the monograph of L.Saulis and V.Statulevicius (1991, p.42).

Next, we require that everywhere for r.v.'s $\xi_j^{(n)}$, $j = \overline{1, n}$ there exist the density $p_{\xi_j^{(n)}}(x)$ and

$$\sup_x p_{\xi_j^{(n)}}(x) \le C_j^{(n)} < \infty. \tag{\textbf{D}}$$

In case where the density does not exist for r.v.'s $\xi_j^{(n)}$, we assume that $C_j^{(n)} = \infty$.

P R O P O S I T I O N 1 . *If a sequence of r.v.'s series $\xi_j^{(n)}, j = \overline{1, n}$, satisfy the condition (B_γ), than for the k^{th} - order cumulant $\Gamma_k(Z_n)$ of a r.v. Z_n, the estimate*

$$|\Gamma_k(Z_n)| \le \frac{(k!)^{1+\gamma}}{\Delta_n^{k-2}}, \qquad k = 3, 4, \ldots \tag{8}$$

is valid, where

$$\Delta_n = \frac{B_n}{K_n}, \qquad K_n := 2 \max_{1 \le j \le n} (K_j^{(n)} \vee \sigma_j^{(n)}). \tag{9}$$

Further we require that $\Delta_n \to \infty$, as $n \to \infty$.

Proof. Note that $\mathbf{E}Z_n = 0$, $\mathbf{D}Z_n = 1$. Taking into consideration that r.v.'s $\xi_j^{(n)}$, $j = \overline{1, n}$, are independent, we obtain

$$f_{Z_n}(t) = f_{S_n}(t/B_n) = \prod_{j=1}^n f_{\xi_j^{(n)}}(t/B_n), \tag{10}$$

and

$$\Gamma_k(S_n) = \sum_{j=1}^n \Gamma_k(\xi_j^{(n)}). \tag{11}$$

Now, making use of condition $(\mathbf{B}_\gamma)$ and basing on inequality (7), we get

$$|\Gamma_k(S_n)| \le \sum_{j=1}^n |\Gamma_k(\xi_j^{(n)})| \le (k!)^{1+\gamma} K_n^{k-2} B_n^2, \tag{12}$$

$k = 3, 4, \ldots$. It is easy to see that $\Gamma_k(Z_n) = \Gamma_k(S_n)/B_n^k$. By virtue of that inequality (8).

Let r.v. $\xi_j^{(n)}(h)$ be conjugate to r.v. $\xi_j^{(n)}$, $j = \overline{1,n}$ with the density function

$$p_{\xi_j^{(n)}(h)}(y) := e^{hy} p_{\xi_j^{(n)}}(y) \Big/ \int_{-\infty}^{\infty} e^{hy} p_{\xi_j^{(n)}}(y) dy. \tag{13}$$

Denote

$$S_n(h) = \sum_{j=1}^{n} \xi_j^{(n)}(h), \qquad Z_n(h) = \frac{(S_n(h) - M_n(h))}{B_n(h)}, \tag{14}$$

$$M_n(h) = \mathbf{E} S_n(h), \qquad B_n^2(h) = \mathbf{D} S_n(h). \tag{15}$$

It is easy to get convinced that

$$M_n(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \Gamma_k(S_n) h^{k-1}, \tag{16}$$

$$B_n^2(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \Gamma_k(S_n) h^{k-2}, \tag{17}$$

where the quantity $h = h(x)$ is the solution of the equation

$$x = \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \Gamma_k(S_n) h^{k-1}. \tag{18}$$

Let $f_{Z_n(h)}(t)$ be the characteristic function of the r.v. $Z_n(h)$. If the r.v.'s $\xi_j^{(n)}$, $j = \overline{1,n}$ are independent, then

$$\begin{aligned} f_{Z_n(h)}(t) \quad &:= \quad \mathbf{E} e^{itZ_n(h)} = \exp\left\{ -it \frac{M_n(h)}{B_n(h)} \right\} \\ &\times \quad \prod_{j=1}^{n} f_{\xi_j^{(n)}(h)} \Big(\frac{t}{B_n(h)} \Big). \end{aligned} \tag{19}$$

Let us denote

$$R_{n,\gamma} = \int_{T_{n,\gamma}}^{T_n} |f_{n,\gamma}^*(t)| \frac{dt}{t}, \qquad T_n \geq T_{n,\gamma}, \tag{20}$$

where $T_{n,\gamma}$ defined by equality (24) and

$$f^*_{n,\gamma}(t) = \begin{cases} \sum_{k=0}^{s} \left(\frac{3}{2}\right)^k \frac{x^k}{k!} f^{(k)}_{Z_n}(t), & \gamma > 0, \\ f_{Z_n(h)}(t), & \gamma = 0. \end{cases} \tag{21}$$

Here

$$s := 2[(1/2)(\Delta_n^2/18)^{1/(1+2\gamma)}] - 2, \tag{22}$$

where Δ_n is defined by equality (9) and $[a]$ is the integral part of number a.

While considering the asymptotic expansions of the distribution function $1 - F_{Z_n}(x) = \mathbf{P}(Z_n \geq x)$ of the r.v. Z_n, the quantities

$$\Delta_{n,\gamma} := c_\gamma \Delta_n^{1/(1+2\gamma)}, \qquad c_\gamma = (1/6)(\sqrt{2}/6)^{1/(1+2\gamma)}, \tag{23}$$

$$T_{n,\gamma} := (3/8)(1 - x/\Delta_{n,\gamma})\Delta_{n,\gamma}, \qquad 0 \leq x < \Delta_{n,\gamma} \tag{24}$$

play an important part.

PROPOSITION 2 . *If r.v.'s $\xi_j^{(n)}$, $j = \overline{1,n}$, with $\mathbf{E}\xi_j^{(n)} = 0$ and ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2 > 0$, $j = \overline{1,n}$ fulfil the condition (B_γ), then for any integer l, $l \geq 3$ in the interval*

$$0 \leq x < \Delta_{n,\gamma},$$

the equality

$$\begin{aligned} \frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} &= \exp\{L^*_{n,m}(x)\}\Bigg\{ \frac{\psi(x)}{\psi(u_n(x))}\Bigg(1 + \sum_{\nu=1}^{l-3} L_{\nu,n}(u_n(x))\Bigg) \\ &+ \theta_1(x+1)\Bigg(\frac{c(l,\gamma,x)}{\Delta_n^{l-2}} + \frac{285\Delta_n}{(1 - x/\Delta_{n,\gamma})} \\ &\times \exp\{-(1 - x/\Delta_{n,\gamma})\sqrt{\Delta_{n,\gamma}}\} + \frac{6q}{T_n} + R_{n,\gamma}\Bigg)\Bigg\} \end{aligned} \tag{25}$$

holds. Here

$$L_{n,m}(x) = \sum_{3 \leq k < m} \lambda_{k,n} x^k, \qquad m = \begin{cases} (1/\gamma) + l - 1, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases} \tag{26}$$

For the coefficients $\lambda_{k,n}$, the estimate

$$|\lambda_{k,n}| \le (2/k)(16/\Delta_n)^{k-2}((k+1)!)^{\gamma}, \qquad k = 3, 4, \ldots \tag{27}$$

is valid. In particular:

$$\begin{aligned}
&\lambda_{3,n} = (1/3)\Gamma_3(Z_n),\\
&\lambda_{4,n} = \Big(1/24\Big)\Big(\Gamma_4(Z_n) - 3\Gamma_3^2(Z_n)\Big),\\
&\lambda_{5,n} = \Big(1/120\Big)\Big(\Gamma_5(Z_n) - 10\Gamma_3(Z_n)\Gamma_4(Z_n) + 15\Gamma_3^2(Z_n)\Big), \ldots .
\end{aligned}$$

For polynomials $L_{\nu,n}(u_n(x))$ an analytical expression has been obtained. In particular:

$$\begin{aligned}
&L_{1,n}(u_n(x)) = -\frac{1}{2}\Gamma_3(Z_n)\frac{1}{x} + \frac{3}{2}\Big(2\Gamma_4(Z_n) - 3\Gamma_3^2(Z_n)\Big)\\
&+\frac{1}{48}\Big(72\Gamma_5(Z_n) - 394\Gamma_3(Z_n)\Gamma_4(Z_n) + 267\Gamma_3^3(Z_n)\Big)x + \ldots,\\
&L_{2,n}(u_n(x)) = \frac{1}{24}\Big(3\Gamma_4(Z_n) - 5\Gamma_3^2(Z_n)\Big) + \frac{1}{24}\Big(3\Gamma_5(Z_n)\\
&-16\Gamma_3(Z_n)\Gamma_4(Z_n) + 15\Gamma_3^3(Z_n)\Big)x + \ldots,
\end{aligned}$$

the quantity

$$u_n(x) = x\Big(1 + \sum_{k=1}^{l-3} c_{k,n}x^k + \theta_2 c^*(l)(x/\Delta_n)^{l-2}\Big), \tag{28}$$

where $c^*(l,\gamma) = 736l(l-1)(7/2)^{l-2}(l!)^{\gamma}$. The quantities $c(l,\gamma,x)$ and $q(l,\gamma)$ are expressed in terms of l, γ and x. Reminder term $R_{n,\gamma}$ defined by equality (20).

The proof can be obtained employing the general lemma of L.Saulis (1996) and the method of cumulants .

Now, basing on Proposition 2, we find the estimates of the remainder term $R_{n,\gamma}$,

$$R_{n,\gamma} = \int_{T_{n,\gamma}}^{T_n} |f^*_{n,\gamma}(t)|\frac{dt}{t}, \tag{29}$$

of asymptotic expansion (25) of probability $\mathbf{P}(Z_n \ge x) = 1 - F_{Z_n}(x)$ in the Cramer zone, as $\gamma = 0$, and in the power Linnik zone, as $\gamma > 0$. This result can be attained, based on V.Statulevicius's (1966) estimates of characteristic functions, as the condition $(\mathbf{D})$ is fulfilled.

In the case where $\gamma = 0$, we denote the quantities $\Delta_{n,\gamma}$ and $T_{n,\gamma}$ defined by equalities (23) and (24), respectively, as follows

$$\begin{aligned} &\Delta_{n,0} = c_0 \Delta_n, \qquad c_0 = (1/6)(\sqrt{2}/6), \\ &T_{n,0} = (1/8)(1 - x/\Delta_{n,0})\Delta_{n,0}, \end{aligned} \tag{30}$$

where

$$\Delta_n = \frac{B_n}{K_n}, \qquad K_n := 2 \max_{1 \le j \le n} (K_j^{(n)} \vee \sigma_j^{(n)}).$$

T H E O R E M 1 . *If r.v.'s* $\xi_j^{(n)}$ *with* $\mathbf{E}\xi_j^{(n)} = 0$ *and* $\sigma_j^{(n)^2} = \mathbf{E}\xi_j^{(n)^2} > 0$, $j = \overline{1,n}$, *satisfy conditions* $(\mathbf{B}_\gamma)$, *as* $\gamma = 0$ *and* $(\mathbf{D})$, *then for all* x,

$$0 \le x < \Delta_{n,0}, \qquad (in \;\; the \;\; Cramer \;\; zone)$$

asymptotic expansion (24) *with remainder term estimate*

$$\begin{aligned} R_{n,0} \;\; &\le \;\; 684 e^4 \pi \sqrt{2\pi} K_n \max_{1 \le r_i \le n} \prod_{i=1}^{4} C_{r_i}^{(n)^{1/4}} \exp\Big\{ -\frac{1}{24 K_n^2} \sum_{j=1}^{n} C_k^{(n)^{-2}} \Big\} \\ &+ \;\; \frac{\pi^2}{2T_{n,0}} \exp\Big\{ -\frac{1}{\pi^2} T_{n,0}^2 \Big\} \end{aligned} \tag{31}$$

is true.

T H E O R E M 2 . *Let r.v.'s* $\xi_j^{(n)}$, $j = 1, 2, \dots$ *with* $\mathbf{E}\xi_j^{(n)} = 0$ *and* $\sigma_j^{(n)^2} = \mathbf{E}\xi_j^{(n)^2} > 0$, $j = \overline{1,n}$, *fulfil conditions* $(\mathbf{D})$, $(\mathbf{B}_\gamma)$, *as* $\gamma > 0$, *and*

$$\underline{\lim}_{n \to \infty} \frac{1}{(1 \vee L_{1,n})\Delta_{n,\gamma}} \frac{1}{K_n^2} \sum_{j=1}^{n} C_j^{(n)^{-2}} \ge d > 0,$$

when $L_{1,n} = \sum_{j=1}^{n} \mathbf{E}|\xi_j^{(n)}|/B_n$. *Then for all* x,

$$0 \le x < \Delta_{n,\gamma}, \qquad (in \;\; the \;\; power \;\; Linnik \;\; zone)$$

asymptotic expansion (25) *with remainder term estimate*

$$R_{n,\gamma} \le c_4(\gamma)(K_n/\Delta_n) \max_{1\le r_i\le n} \prod_{i=1}^{4} C_{r_i}^{(n)^{1/4}} \Big\{ -\frac{c_3}{2K_n^2} \sum_{k=1}^{n} C_k^{(n)^{-2}} \Big\}$$
$$+T_{n,\gamma}^2 \exp\Big\{ -\frac{1}{16\pi^2} T_{n,\gamma}^2 \Big\} + \frac{5\pi^2 x^2}{8} T_{n,\gamma} \exp\Big\{ -\frac{1}{\pi^2} T_{n,\gamma}^2 \Big\} \qquad (32)$$

holds, where K_n, $\Delta_{n,\gamma}$ *and* $T_{n,\gamma}$ *are defined by equalities* (9), (21) *and* (22), *respectively.*

T H E O R E M 3 . *Let a sequence of independent r.v.'s* $\xi_j^{(n)}$, $j = 1, 2, \ldots$ *with means* $\mathbf{E}\xi_j^{(n)} = 0$ *and non - zero finite variances* $\sigma_j^{(n)^2} = \mathbf{E}\xi_j^{(n)^2} > 0$, $j = \overline{1, n}$ *satisfy the generalized Bernstein condition* $(\mathbf{B}_\gamma)$. *Then in the interval*

$$0 \le x \le \Delta_{n,\gamma},$$

the equalities of large deviations

$$\frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} = exp\{L_{n,m}(x)\}\Big(1 + \theta_1 f_1(x)\frac{x+1}{\Delta_{n,\gamma}}\Big),$$
$$\frac{F_{Z_n}(-x)}{\Phi(-x)} = exp\{L_{n,m}(-x)\}\Big(1 + \theta_2 f_2(x)\frac{x+1}{\Delta_{n,\gamma}}\Big), \qquad (33)$$

hold, where

$$f_i(x) = \frac{60\Big(1 + 10\Delta_{n,\gamma}^2 exp\{ -(1 - x/\Delta_{n,\gamma})\sqrt{\Delta_{n,\gamma}}\}\Big)}{1 - x/\Delta_{n,\gamma}},$$

$i = 1, 2$; $L_{n,m}(x)$ *and* $\Delta_{n,\gamma}$ *are defined by equalities* (26) *and* (23), *respectively.*

C O R O L L A R Y 1 . *Let* $\xi_j^{(n)}$, $j = 1, 2, \ldots$ *be a sequence of independent r.v.'s with means* $\mathbf{E}\xi_j^{(n)} = 0$ *and non - zero finite variances* $\sigma_j^2 = \mathbf{E}\xi_j^{(n)^2} > 0$, $j = \overline{1, n}$, *and let it fulfil the generalized Bernstein condition* $(\mathbf{B}_\gamma)$. *Then for all*

$$x = (\Delta_{n,\gamma}^{\tau}), \qquad \tau = \tau(\gamma) = \frac{1}{(1 + 2(1 \vee \gamma)},$$

$\Delta_{n,\gamma} \to \infty$, *as* $n \to \infty$ *the equalities*

$$\lim_{n\to\infty} \frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} = 1, \qquad \lim_{n\to\infty} \frac{F_{Z_n}(-x)}{\Phi(-x)} = 1 \qquad (34)$$

hold.

T H E O R E M 4 *Let* $\xi_j^{(n)}$, $j = 1, 2, \ldots$ - *be a sequence of independent r.v.'s satisfy Bernstein condition* $(\mathbf{B}_\gamma)$. *Then for r.v.* Z_n, *defined by formula* (4), *the exponential inequalities*

$$\mathbf{P}\{\pm Z_n \geq x\} \leq \begin{cases} \exp\Big\{-(1/4)Hx^2\Big\}, & 0 \leq x < (H\Delta_n)^{1/(1+2\gamma)}, \\ \exp\Big\{-(1/4)(x\Delta_n)^{1/(1+\gamma)}\Big\}, & x \geq (H\Delta_n)^{1/(1+\gamma)}, \end{cases} \tag{35}$$

are valid, as $H = 2^{1+\gamma}$. *Here* Δ_n *defined by formula* (9).

"Asymptotic expansions of the distribution density function for the sum of random variables in the series scheme in large deviation zones". This chapter is aimed at obtaining asymptotic expansions of the density function in the large deviation Cramer and power Linnik zones for the sum of independent r.v.'s $\xi_1^{(n)}, \xi_2^{(n)}, \ldots, \xi_j^{(n)}$, $j = 1, 2, \ldots, n$ with means $\mathbf{E}\xi_j^{(n)} = 0$, and variances ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2$ in the series scheme.

The results will be obtained, based on L.Saulis' (1991) generalized lemma for large deviations by using the method of cumulants and the estimates of characteristic functions. In order to apply the general lemma for large deviations, we need the estimate of the k^{th} - order cumulant $\Gamma_k(Z_n)$, $k = 3, 4, \ldots$ of the r.v. Z_n., in case r.v.,'s $\xi_j^{(n)}$, $j = 1, 2, \ldots$, fulfil the generalized Bernstein condition $(\mathbf{B}_\gamma)$. Recall that the densities $p_{\xi_j^{(n)}}$ satisfy condition

$$\sup_x p_{\xi_j^{(n)}(x)} \leq C_j^{(n)} < \infty. \tag{$\mathbf{D}$}$$

P R O P O S I T I O N 3 . *Let a sequence of series* $\xi_j^{(n)}$ *with means* $\mathbf{E}\xi_j^{(n)} = 0$ *and variances* $\sigma_j^2 = \mathbf{E}{\xi_j^{(n)}}^2 > 0$, $j = 1, 2, \ldots$ *satisfy conditions* $(\mathbf{B}_\gamma)$ *and* $(\mathbf{D})$. *Then for each integer* l, $l \geq 1$ *and* $T_n \geq T_{n,\gamma}$, *in the interval*

$$0 \leq x < \Delta_{n,\gamma},$$

the asymptotic expansion

$$\begin{aligned} \frac{p_n(x)}{\varphi(x)} &= \exp\{L_{m,n}(x)\}\biggl(1 + \sum_{\nu=0}^{l-1} M_{\nu,n}(x) + \\ &+ \theta_1 q(\gamma, l)\left(\frac{x+1}{\Delta_n}\right)^l + R^*_{n,\gamma}\biggr), \end{aligned} \tag{36}$$

holds, where

$$R^*_{n,\gamma} = \frac{2}{\pi}\int_{T_{n,\gamma}}^{\infty} |f^*_{n,\gamma}(t)|dt, \tag{37}$$

when $f^*_{n,\gamma}(t)$ *and* $L_{m,n}(x)$ *are defined by formulas (21) and (26), respectively. The polynomials* $M_{\nu,n}$ *are found by the formulas*

$$M_{\nu,n}(x) = \sum_{k=0}^{\nu} K_{k,n}(x)q - \nu - k, n(x), \tag{38}$$

$$K_{\nu,n}(x) = \sum \prod_{m=1}^{\nu} \frac{1}{k_m!}(-\lambda_{m+2,n}x^{m+2})^m, \qquad K_0(x) \equiv 1, \tag{39}$$

$$q_{\nu,n}(x) = \sum H_{\nu+2l,n}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}(\Gamma_{m+2}(Z_n)/(m+2)!)^{k_m}, \tag{40}$$

$$q_0(x) \equiv 1,$$

where $H_m(x)$ *are the* m^{th} *- order Chebyshev - Hermite polynomials*

$$H_m(x) \;\; = \;\; (-1)^m \frac{1}{\varphi(x)} \frac{d^m}{dx^m}\varphi(x).$$

In particular:

$$\begin{aligned} M_{0,n}(x) &\equiv 0 \qquad M_{1,n}(x) = -1/2\Gamma_3(Z_n), \\ M_{2,n}(x) &= 1/8(5\Gamma_3^2(Z_n) - 2\Gamma_4(Z_n))x^2 + 1/24(3\Gamma_4(Z_n) - 5\Gamma_3^2(Z_n)), \\ M_{3,n}(x) &= 1/24(34\Gamma_3(Z_n)\Gamma_4(Z_n) - 4\Gamma_5(Z_n) - 45\Gamma_3^3(Z_n))x^2 \\ &+ 1/48(6\Gamma_5(Z_n) - 35\Gamma(Z_n)\Gamma(Z_n) + 35\Gamma(Z_n))x, \ldots. \end{aligned}$$

The expression of the quantity $q(\gamma, l)$ *is known.*

Let us find the estimates of the remainder term $R^*_{n,\gamma}$, defined by equality (37), of asymptotic expansion (36) in the Cramer zone, as $\gamma = 0$, and the power Linnik zone, as $\gamma > 0$.

T H E O R E M 5 . *Let a sequence of r.v.'s* $\xi_j^{(n)}$ *with* $\mathbf{E}\xi_j^{(n)} = 0$ *and* ${\sigma_j^{(n)}}^2 > 0$, $j = 1, 2, \ldots$ *satisfy conditions* $(\mathbf{D})$ *and* $(\mathbf{B}_\gamma)$, *as* $\gamma = 0$. *Then for all* x

$$0 \leq x < \Delta_{n,0} \qquad (in \;\; the \;\; Cramer \;\; zone)$$

asymptotic expansion (36) holds with the estimate of the remainder term

$$R_{n,0} \leq \frac{\pi^2}{2T_{n,0}^2} \exp\Big\{ -\frac{T_{n,0}^2}{\pi^2}\Big\} + \Big(\frac{C_1 K_n}{\Delta_{n,0}}\Big) \times \max_{1\leq j\leq n} \prod_{i=1}^{4} C_{r_i}^{(n)^{1/4}} \exp\Big\{ -\frac{c_3}{K_n^2}\sum_{j=1}^{n} 1/C_k^{(n)^2}\Big\}, \tag{41}$$

where c_3 is a positive constant.

THEOREM 6 . *Let r.v.'s $\xi_j^{(n)}$, $j = \overline{1,n}$ with $\mathbf{E}\xi_j^{(n)} = 0$ and $\sigma_j^{(n)^2} = \mathbf{E}\xi_j^{(n)^2} > 0$, $j = 1, 2, \ldots$ satisfy conditions $(\mathbf{D})$, $(\mathbf{B}_\gamma)$, as $\gamma > 0$, and*

$$\lim_{n\to\infty} 1/(1 \vee L_{1,n})\Delta_{n,\gamma} \sum_{j=1}^{n} 1/(\sigma_j^{(n)^2} + N_n^2 C_j^{(n)^2}) \geq c_2 > 0.$$

Then for all x,

$$0 \leq x < \Delta_{n,\gamma} \qquad (in \;\; the \;\; power \;\; Linnik \;\; zone)$$

asymptotic expansion (36) holds with the estimate of the remainder term

$$R_{n,\gamma} \leq B_n(N_n/T_{n,\gamma}) \max_{1\leq r_i\leq n} \prod_{i=1}^{4} C_{r_i}^{(n)^{1/4}} \Big(1 + N_n(\sigma_{r_i}^{(n)}/2\sqrt{2})\Big) \times \exp\Big\{ -\sum_{j=1}^{n} \frac{1}{4(\sigma_j^{(n)^2} + N_n^2)C_j^{(n)^2}}\Big\}. \tag{42}$$

Here $N_n \leq 4B_n L_{3,n}$, where $L_{3,n}$ is the third order Lyapunov fraction.

Discounted limit theorems. Let $X_0, X_1, X_2, \ldots$ be a sequence of independent r.v.'s with the same distribution function $F(x)$. Let v be a discounting factor $(0 < v < 1)$. Then, let us define

$$S_v = \sum_{k=0}^{\infty} v^k X_k, \tag{43}$$

Suppose that the first three moments of the r.v. X_k are finite:

$$\mu = \int_{-\infty}^{\infty} x dF(x) \quad < \quad \infty, \qquad \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 dF(x) < \infty,$$

$$\rho \quad = \quad \int_{-\infty}^{\infty} |x-\mu|^3 dF(x) < \infty. \tag{44}$$

Then it is easy to see that the mean and variance of the r.v. S_v are

$$\mathbf{E}S_v = \mu(1-v)^{-1}, \qquad \mathbf{D}S_v = \sigma^2(1-v^2)^{-1}, \tag{45}$$

respectively. Next we shall consider a centered normed r.v.

$$Z_v = \sigma^{-1}(1-v)^{\frac{1}{2}}(S_v - \mu(1-v)^{-1}), \tag{46}$$

with the mean $\mathbf{E}Z_v = 0$ and variance $\mathbf{D}Z_v = (1+v)^{-1}$. Let us denote the distribution function of the r.v. Z_v as $F_v(x)$, and consider a normal distribution with zero mean and variance $(1+v)^{-1}$,

$$N_v(x) = \left(\frac{1+v}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{x} \exp\Big\{-\frac{1+v}{2}\Big\} y^2 dy \tag{47}$$

In his work Hans U.Gerber has proved the Berry - Esseen theorem in the case where discounting is applied to a r.v.: if equalities (44) hold, then for all x the inequality

$$|F_v(x) - N_v(x)| \le 5.4(\rho/\sigma^3)(1-v)^{\frac{1}{2}} \tag{48}$$

is true.

We shall consider the asymptotic expansion for the probability $\mathbf{P}(Z_v \ge x)$, as $x = x_v \to \infty$, $v \to 1$, i.e., we shall prove large deviation theorems for the r.v. Z_v defined by equality (46), using the method of cumulants when the centred moments $\mathbf{E}(X_0-\mu)^s$ of the r.v. X_0 satisfy the generalized Bernstein condition $(\widehat{\mathbf{B}}_\gamma)$:

let there exist quantities $\gamma \ge 0$ and $K > 0$ such that

$$|\mathbf{E}(X_0-\mu)^s| \le (s!)^{1+\gamma} K^{s-2}\sigma^2, s = 3, 4, \dots . \tag{$\widehat{\mathbf{B}}_\gamma$}$$

Put

$$\Delta_v = \frac{\sigma}{2(K \vee \sigma)}(1-v)^{-\frac{1}{2}}, \qquad \Delta_v(\gamma) = c_v(\gamma)\Delta_v^{\frac{1}{1+2\gamma}}, \tag{49}$$

where $c_v(\gamma) = (1/6)(\sqrt{2}/(6(1+v)^{1+\gamma}))^{1/(1+2\gamma)}$ and $a \vee b = \max\{a, b\}$.

T H E O R E M 7 . *Let a uniformly distributed r.v. X_k with $\mathbf{E}X_k = \mu$ and $\sigma^2 = \mathbf{E}(X-\mu)^2$, $k = 0, 1, 2, \ldots$ satisfy the condition $(\widehat{B}_\gamma)$. Then, for the distribution function $F_v(x)$ of the r.v. Z_v,defined by equality (46), in the interval $0 \le x < \Delta_v(\gamma)$ the large deviation equalities*

$$\begin{aligned} \frac{1-F_v(x)}{1-N_v(x)} &= \exp\{L_\gamma(x)\}\left(1+\theta_1 f(x)\frac{x+1}{\Delta_v(\gamma)}\right), \\ \frac{F_v(-x)}{N_v(-x)} &= \exp\{L_\gamma(-x)\}\left(1+\theta_2 f(x)\frac{x+1}{\Delta_v(\gamma)}\right) \end{aligned} \tag{50}$$

are valid. Here

$$\begin{aligned} f(x) &= \frac{60\Big(1+(1+v)\Delta_v^2(\gamma)\exp\Big\{-(1-x\Delta_v(\gamma))\sqrt{\Delta_v(\gamma)}\Big\}\Big)}{(1-x/\Delta_v(\gamma))}, \\ L_\gamma(x) &= \sum_{3\le k<p} \lambda_k x^k + \theta(x/\Delta_v(\gamma)), \\ p &= \begin{cases} (1/\gamma)+2, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases} \end{aligned} \tag{51}$$

The inequality

$$|\lambda_k| \le \frac{2(1+v)}{k}\left(\frac{16(1+v)}{\Delta_v}\right)^{k-2}((k+1)!)^\gamma, \qquad k = 3, 4, \ldots, \tag{52}$$

holds for the coefficients λ_k, besides

$$L_\gamma(x) \le \frac{(1+v)x^2}{2}\frac{x}{x+8\Delta_v(\gamma)}, \qquad L_\gamma(-x) \ge -\frac{(1+v)x^3}{3\Delta_v(\gamma)}. \tag{53}$$

T H E O R E M 8 . *Let the r.v.'s X_k, $k = 0, 1, 2, \ldots$ satisfy the condition $(\widehat{B}_\gamma)$. Then, for distribution function $F_v(x)$ with all $x \ge 0$, $x = o(\Delta_v^\nu)$, where Δ_v is defined by equality (49) and $\nu = \nu(\gamma) = (1+2\max\{1,\gamma\})^{-1}$, the equalities*

$$\lim_{v\to 1}\frac{1-F_v(x)}{1-N_v(x)} = 1, \qquad \lim_{v\to 1}\frac{F_v(-x)}{N_v(-x)} = 1 \tag{54}$$

are true. In a special case, where $\gamma = 0$, relations (54) hold for $x \ge 0$, $x = o((1-v)^{-\frac{1}{6}})$.

T H E O R E M 9 . *Let the r.v.'s* X_k, $k = 0, 1, 2, \ldots$ *satisfy the condition* $(\widehat{B}_\gamma)$. *Then, for the probability* $\mathbf{P}(\pm Z_v \geq x)$ *with* $H_v = 2^{1+\gamma}(1+v+v^2)^{-1}$ *and* $\Delta_v = \frac{\sigma}{2(K \vee \sigma)}(1-v)^{-\frac{1}{2}}$ *of the r.v.* Z_v *the exponential inequalities*

$$\mathbf{P}(\pm Z_v \geq x) \leq \begin{cases} e^{-\frac{1}{4H_v}x^2}, & 0 \leq x \leq (H_v^{1+\gamma}\Delta_v)^{1/(1+2\gamma)}, \\ e^{-\frac{1}{4}\left(x\Delta_v\right)^{1/(1+\gamma)}}, & x \geq (H_v^{1+\gamma}\Delta_v)^{1/(1+2\gamma)} \end{cases} \tag{55}$$

are valid.

Summation of random variables with weights. Let $X_1, X_2, \ldots, X_n$ be independent, non - identically distributed r.v. with the mean that, without loss of generality, is supposed be equal to zero, i.e., $\mathbf{E}X_j = 0$, positive variance $\sigma_j^2 = \mathbf{E}X_j^2 > 0$ and a_{nj} are non - negative constants.

Denote:

$$\xi_j^{(n)} = a_{nj}X_j, \qquad j = 1, 2, \ldots, n, \tag{56}$$

then

$$\tilde{S}_n = \sum_{j=1}^{n} a_{nj}X_j, \qquad \tilde{B}_n^2 = \sum_{j=1}^{n} a_{nj}^2\sigma_j^2, \qquad \tilde{Z}_n = \frac{\tilde{S}_n}{\tilde{B}_n}, \tag{57}$$

$$F_{\tilde{Z}_n}(x) = \mathbf{P}(\tilde{Z}_n < x), \qquad p_{\tilde{Z}_n}(x) = \frac{d}{dx}F_{\tilde{Z}_n}(x), \tag{58}$$

P R O P O S I T I O N 4 . *Let independent, non - identically distributed r.v.'s* X_j, $j = 1, 2, \ldots$ *with* $\mathbf{E}X_j = 0$ *and* $\sigma_j^2 = \mathbf{E}X_j^2$ *satisfy the condition* $(\tilde{\mathbf{B}}_\gamma)$*: there exist quantities* $\gamma \geq 0$ *and* $K > 0$ *such that*

$$|\mathbf{E}X_j^k| \leq (k!)^{1+\gamma}K^{k-2}\sigma_j^2, \qquad k = 3, 4, \ldots \ . \tag{$\tilde{B}_\gamma$}$$

Then for the k^{th} *- order cumulant* $\Gamma_k(\tilde{Z}_n)$ *of the r.v.* $\tilde{Z}_n$*, the estimate*

$$|\Gamma_k(\tilde{Z}_n)| \leq \frac{(k!)^{1+\gamma}}{\tilde{\Delta}_n^{k-2}}, \qquad k = 3, 4, \ldots, \tag{$\tilde{S}_\gamma$}$$

holds, where

$$\tilde{\Delta}_n = \frac{\tilde{B}_n}{\tilde{K}_n}, \qquad \tilde{K}_n = \max_{1 \leq j \leq n} (2a_{nj}\{K \vee \sigma_j\}). \tag{59}$$

P R O P O S I T I O N 5 *. Let, for the r.v.'s* X_j, $j = \overline{1, n}$ *with the mean* $\mathbf{E}X_j = 0$ *and variance* $\sigma_j = \mathbf{E}X_j^2 > 0$, *there exist the distribution density* $p_{X_j}(x)$, *and* $\sup_x p_{X_j}(x) < C$, *then the density* $p_{\xi_j^{(n)}}(x)$ *of the r.v.* $\xi_j^{(n)} = a_{nj}X_j$ *satisfies the inequality*

$$\sup_x p_{\xi_j^{(n)}}(x) \le \frac{C_j}{a_{nj}}. \qquad (\tilde{\mathbf{D}})$$

Thus, in case the density of the r.v. $\xi_j^{(n)}$ does not exist, we assume that $C_j = \infty$.

Basing on Proposition 2, as $Z_n = \tilde{Z}_n$, and requiring that the condition $(\tilde{\mathbf{B}}_\gamma)$ be fulfilled, we find the estimates of the remainder term

$$R_{n,\gamma} = \int_{\tilde{T}_{n,\gamma}}^{T_n} |f_{n,\gamma}^*(t)| \frac{dt}{t}$$

of the asymptotic expansion for the probability $\mathbf{P}(\tilde{Z}_n \ge x) = 1 - F_{\tilde{Z}_n}(x)$ with $\gamma = 0$ (in the Cramer zone), and with $\gamma > 0$ (in power Linnik zone). Here

$$\tilde{T}_{n,\gamma} = (3/8)(1 - x/\tilde{\Delta}_{n,\gamma})\tilde{\Delta}_{n,\gamma}, \qquad \tilde{\Delta}_{n,\gamma} = \Delta_{n,\gamma} = c_\gamma \Delta_n^{1/(1+2\gamma)},$$
$$c_\gamma = (1/6)(\sqrt{2}/6)^{1/(1+2\gamma)}, \qquad \Delta_n = \tilde{\Delta}_n,$$

$\tilde{\Delta}_n$ is defined by equality (59).

T H E O R E M 10 *Let independent non - identically distributed r.v.'s* X_j *with mean* $\mathbf{E}X_j = 0$ *and variance* $\sigma_j^2 = \mathbf{E}X_j^2 > 0$ *satisfy conditions* $(\tilde{\mathbf{B}}_\gamma)$ *and* $(\tilde{\mathbf{D}})$, *then for the r.v.* $Z_n = \tilde{Z}_n$ *asymptotic expansions* (25) *and* (36) *hold with the remainder term estimates* (31) *and* (32), *and* (41) *and* (42), *respectively, with*

$$C_j^{(n)} = \frac{C_j}{a_{nj}}, \qquad \Delta_n = \tilde{\Delta}_n = \frac{\tilde{B}_n}{\tilde{K}_n},$$
$$K_n = \tilde{K}_n = \max_{1 \le j \le n} (2a_{nj}\{K \vee \sigma\}).$$

CONCLUSIONS

• Asymptotic expansion and optimal estimation of the reminder for the normed sums of independent non - identically distributed random variables, in a triangular array scheme, satisfying the generalized Bernstein's condition, are obtained.

• Asymptotic expansion for density of the normed sum of independent non - identically distributed random variables is obtained. Theorems enabling to obtain the reminder term in Cramer and power Linnik zones, are proved.

• The results are obtained joining the methods of characteristic functions and cumulants.

• The discounted theorems of large deviations (without asymptotic expansion) and exponential inequalities are proved by cumulant methods.

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Author's CV.
1975-11-27. I was born in Alytus.
1983-1993. Alytus secondary school No9 (Secondary School "Jotvingiai").
1993-1997. Vilnius Gediminas Technical University, Faculty of Fundamental Studies. Bachelor of Engineering Informatics science.
1997-1999. Vilnius Gediminas Technical University, Faculty of Fundamental Studies. Master of Mathematical science.
1999-2003. Vilnius Gediminas Technical University, Faculty of Fundamental Studies. PhD studentship.

LIST OF PUBLICATIONS

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• D. Deltuvienė, L. Saulis (2003). Asymptotic Expansions of the Distribution Density in the Large Deviations Zones for Sums of Independent Random Variables in the Series Scheme. Acta Applicandae Mathematicae, **78**, Dortrecht, Kluwer Academic Publishers B.V., p. 87 - 97.

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Scientific approval and contribution of the results

The main results of this doctoral dissertation are published in seven scientific papers. The contributed reports were given at seven conferences:

• D. Deltuvienė, L. Saulis. Nonuniform estimate of the discounted limit theorem // 45-th conference of Lithuanian Mathematical Society, June 17 - 18, 2004, Kaunas, Lithuania.

• D. Deltuvienė, L. Saulis. Didžiųjų nuokrypių teoremos atsitiktinių dydžių Abelio sumoms // 44-th conference of Lithuanian Mathematical Society, June 19 - 20, 2003, Vilnius, Lithuania.

• D. Deltuvienė, L. Saulis. Asymptotic expansions of the distribution densities in the large deviations zones for sums of independent random variables in the series scheme // 8-th International Vilnius Conference on Probability Theory and Mathematical Statistics, June 23 - 29, Vilnius, Lithuania.

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ASIMPTOTINIAI SKLEIDINIAI DIDŽIŲJŲ NUOKRYPIŲ ZONOSE

Daktaro disertacijos santrauka

Dovilė Deltuvienė

Tikimybių teorijos ir matematinės statistikos ribinių teoremų didžiųjų nuokrypių problematikoje didelė darbų dalis tenka nepriklausomų ir priklausomų atsitiktinių dydžių sumų $S_n = X_1 + X_2 + \ldots + X_n$ skirstinių asimptotinei analizei. Čia itin svarbią vietą užima eksponentinių momentų egzistavimo sąlyga: egzistuoja dydžiai $a > 0$ ir $\gamma \geq 0$, tokie, kad $\mathbf{E}\exp\left\{a|X_j|^{\frac{1}{1+\gamma}}\right\} < \infty$, $j = 1, 2, \ldots, n$. Tuo atveju, kai $\gamma = 0$, tai sakoma, kad atsitiktinis dydis X_j tenkina Kramerio sąlygą. Šiuo atveju atsitiktinio dydžio X_j charakteristinė funkcija $f_{X_j}(t) = \mathbf{E}\exp\{itX_j\}$ yra analizinė taško $t = 0$ aplinkoje. Kai atsitiktiniai dydžiai X_j, $j = 1, 2, \ldots, n$ tenkina minėtą sąlygą su $\gamma > 0$, tai sakoma, kad jie tenkina Liniko sąlygą. Šiuo atveju, egzistuoja atsitiktinio dydžio X_j visų eilių momentai, bet jų augimas neužtikrina charakteristinės funkcijos $f_{X_j}(t)$ analiziškumo nulinio taško aplinkoje.

Kai sumos S_n atskiri dėmenys tenkina eksponentinių momentų sąlygą, tuomet H. Kramerio, V. Petrovo, J. Liniko, S.V. Nagajevo darbuose buvo gautos didžiųjų nuokrypių teoremos, kaip taisyklė vienodai pasiskirsčiusiems atsitiktiniams dydžiams. Šie rezultatai buvo gauti sudėtingu analitiniu metodu.

Įvairių statistikų didžiųjų nuokrypių tikimybėms nagrinėti V.Statulevičius (1966) pasiūlė kumulantų (semiinvariantų) metodą, reikalaudamas, kad bet koks atsitiktinis dydis X su vidurkiu $\mathbf{E}X = 0$ ir dispersija $\mathbf{D}X = 1$ tenkintų sąlygą: egzistuoja dydžiai $\gamma \geq 0$ ir $\Delta > 0$, tokie, kad

$$|\Gamma_k(X)| \leq \frac{(k!)^{1+\gamma}}{\Delta^{k-2}}, \qquad k = 3, 4, \ldots, \qquad (S_\gamma)$$

kur $\Gamma_k(X)$ - atsitiktinio dydžio X, $k - tosios$ eilės kumulantas

$$\Gamma_k(X) = \frac{1}{i^k}\frac{d^k}{dt^k}\ln f_X(t)\bigg|_{t=0}, \qquad k = 1, 2, \ldots .$$

Ši sąlyga yra lengvai patikrinama įvairioms statistikoms. Kumulantų elgesio tyrimas yra daug paprastesnis nei pasiskirstymo funkcijų uodegų analizė.

Darbo aktualumas ir mokslinis naujumas. Šis darbas skirtas atsitiktinių dydžių serijų schemoje normuotos sumos pasiskirstymo ir jo tankio funkcijų asimptotinių skleidinių gavimui didžiųjų nuokrypių Kramerio ir laipsninėse Liniko zonose, kai sumos $S_n = \sum_{j=1}^n \xi_j^{(n)}$ atskiri dėmenys $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$, su vidurkiu $\mathbf{E}\xi_j^{(n)} = 0$ ir dispersijomis ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2 > 0$, tenkina apibendrintą S.N.Bernšteino sąlygą: egzistuoja dydžiai $\gamma \geq 0$ ir $K_j^{(n)}$ tokie, kad

$$\left|\mathbf{E}\Big(\xi_j^{(n)}\Big)^k\right| \leq (k!)^{1+\gamma}(K_j^{(n)})^{k-2}{\sigma_j^{(n)}}^2, \qquad k = 3, 4, \ldots . \qquad (\mathbf{B}_\gamma)$$

Tai gana sudėtingas uždavinys, kuris tikimybių teorijos ir matematinės statistikos ribinių teoremų didžiųjų nuokrypių problematikoje sprendžiamas pirmą kartą. Jo naujumas ir originalumas gl$\overline{\text{u}}$di tame, kad asimptotinių skleidinių su optimaliais liekamųjų narių įverčiais didžiųjų nuokrypių zonose gavimui, be kumulantų metodo reikia panaudoti ir klasikinį charakteristinių funkcijų metodą. Be to, sprendžiant darbe suformuluotus uždavinius, skirtingai nuo gerai žinomų rezultatų tikimybių teorijos ir matematinės statistikos ribinių teoremų problematikoje reikia vertinti konstantas. Tai dažnai techniškai gana sudėtingas uždavinys.

Darbe gauti rezultatai turi dideles pritaikymo galimybes tikimybių teorijoje, matematinėje statistikoje, ekonometrijoje ir t.t. Tai demonstruojama paskutiniame disertacijos skyriuje, kuriame įrodytos didžiųjų nuokrypių teoremos atsitiktinių dydžių sumavime su svoriais ir diskontavimo ribinės teoremos.

Disertacijos pagrindiniai rezultatai yra 1 - 2 teoremos, bei 5 - 9 teoremos , kurių teiginiai gaunami dviem etapais. Esant patenkintai sąlygai $(\mathbf{B}_\gamma)$ parodome, kad atsitiktinis dydis

$$Z_n = B_n^{-1}S_n, \qquad S_n = \sum_{j=1}^n \xi_j^{(n)}, \qquad B_n^2 = \sum_{j=1}^n {\sigma_j^{(n)}}^2,$$

$k - tosios$ eilės kumulantas $\Gamma_k(Z_n)$ tenkina nelygybę

$$|\Gamma_k(Z_n)| \leq (k!)^{1+\gamma}\Delta_n^{2-k}, \qquad k = 3, 4, \ldots,$$

kur

$$\Delta_n = K_n^{-1}B_n, \qquad K_n := 2\max_{1\leq j\leq n}\Big(K_j^{(n)} \vee \sigma_j^{(n)}\Big).$$

Čia $a \vee b = \max\{a, b\}$.

Antrasis etapas, remiantis klasikiniu charakteristinių funkcijų metodu, bei žinomais (V.Statulevičius) charakteristinių funkcijų įverčiais, gaunami at.d. Z_n skirstinio asimptotinio skleidinio ir skirstinio tankio asimptotinio skleidinio liekamieji nariai.

Darbo tikslai. Disertacijos tikslas, remiantis bendrosiomis didžiųjų nuokrypių lemomis (L.Saulis (1997), (1989)) gauti tikimybės $\mathbf{P}(Z_n \geq x)$ asimptotinius skleidinius didžiųjų nuokrypių zonose su liekamojo nario

$$R_{n,\gamma} = \int_{T_{n,\gamma}}^{T_n} |f^*_{n,\gamma}(t)| \frac{dt}{t},$$

įverčiais, kai

$$T_{n,\gamma} := (3/8)(1 - x/\Delta_{n,\gamma})\Delta_{n,\gamma}.$$
$$\Delta_{n,\gamma} := c_\gamma \Delta_n^{1/(1+2\gamma)}, \qquad c_\gamma = (1/6)(\sqrt{2}/6)^{1/(1+2\gamma)},$$

ir

$$f^*_{n,\gamma}(t) = \begin{cases} \sum_{k=0}^{s} \left(\frac{3}{2}\right)^k \frac{x^k}{k!} f^{(k)}_{Z_n}(t), & \gamma > 0, \\ f_{Z_n(h)}(t), & \gamma = 0, \end{cases}$$

kai $\gamma = 0$ (Kramerio zonoje) ir, kai $\gamma > 0$ (laipsninėse Liniko zonose). Integralo $R_{n,\gamma}$ įvertinimui panaudojami V.Statulevičiaus (1965) bet kokio atsitiktinio dydžio ξ charakteristinių funkcijų įverčiai. Čia susiduriama su papildomais sunkumais, nes reikia įvertinti ne tik atsitiktinio dydžio Z_n charakteristinę funkciją $f_{Z_n}(t)$, bet ir jos išvestines. Kai $\gamma = 0$ reikia vertinti atsitiktinio dydžio $Z_n(h)$, kuris yra sujungtinis atsitiktiniam dydžiui Z_n, charakteristinę funkciją $f_{Z_n(h)}(t)$. Atsitiktinio dydžio $Z_n(h)$ charakteristinė funkcija apibrėžta formule (19), 11 psl.

Disertacijos struktūra Disertacija sudaro įvadas ir penki skyriai, išvados. Disertacijos apimtis - 88 puslapiai. Disertacija parašyta lietuvių kalba.

Pirmajame skyriuje **"Tyrimų apžvalga"** aptariama, kas buvo padaryta disertacijos tema Lietuvoje ir užsienyje, kitų autorių ir disertacijos autoriaus atlikti darbai, nagrinėjant šią problematiką.

Antrajame skyriuje **"Bendrosios lemos"**, suformuluotos didžiųjų nuokrypių bendrosios lemos bet kokiam atsitiktiniam dydžiui ξ, su vidurkiu $\mathbf{E}\xi = 0$, vienetine dispersija $\mathbf{D}\xi = 1$ ir pasiskirstymo funkcija $F_\xi(x) = \mathbf{P}(\xi < x)$, kai šis atsitiktinis dydis tenkina V.Statulevičiaus sąlygą $\mathbf{S}_\gamma$. Remiantis šiomis bendrosiomis lemomis, gausime atsitiktinio dydžio Z_n skirstinio ir jo tankio funkcijų asimptotinius skleidinius didžiųjų nuokrypių zonose su liekamųjų narių įverčiais.

Trečiajame skyriuje **"Atsitiktinių dydžių serijų schemoje sumos pasiskirstymo funkcijos asimptotiniai skleidiniai didžiųjų nuokrypių Kramerio ir laipsninėse Liniko zonose"** gautos ir įrodytos teoremos nevienodai pasiskirsčiusių atsitiktinių dydžių sumos $S_n = \sum_{j=1}^n \xi_j^{(n)}$ serijų schemoje, kai atskiri sumos dėmenys tenkina apibendrintą Bernšteino sąlygą $\mathbf{B}_\gamma$, pasiskirstymo funkcijai Kramerio

$(\gamma = 0)$ bei laipsninėse Liniko zonose $(\gamma > 0)$. Teoremų įrodymai gauti remiantis charakteristinių funkcijų metodu.

Ketvirtajame skyriuje **"Atsitiktinių dydžių serijų schemoje sumos skirstinio tankio funkcijos asimptotiniai skleidiniai didžiųjų nuokrypių zonose"**, remiantis kumulantų metodu gautas atsitiktinio dydžio Z_n skirstinio tankio funkcijos asimptotinis skleidinys, bei šio skleidinio liekamojo nario įverčiai Kramerio bei laipsninėse Liniko zonose.

Penktajame skyriuje **"Bendrųjų teoremų taikymai"** įrodytos teoremos konkretiems atvejams: kai atsitiktiniai dydžiai sumuojami su svoriais ir diskontavimo ribinės teoremos.

Išvados.

• Gautas nepriklausomų nevienodai pasiskirsčiusių atsitiktinių dydžių normuotos sumos serijų schemoje, skirstinio asimptotinis skleidinys su optimaliu liekamojo nario įverčiu didžiųjų nuokrypių Kramerio ir laipsninėse Liniko zonose, kai atskiri dėmenys tenkina apibendrintą S.N.Bernšteino sąlygą.

• Gautas nepriklausomų nevienodai pasiskirsčiusių atsitiktinių dydžių normuotos sumos pasiskirstymo tankio asomptotinis skleidinys. Įrodytos teoremos šio skleidinio liekamojo nario gavimui Kramerio ir laipsninėse Liniko zonose.

• Rezultatai gauti apjungiant kumulantų ir charakteristinių funkcijų metodus.

• Įrodytos diskontavimo didžiųjų nuokrypių teoremos (be asimptotinio skleidinio) bei eksponentinės nelygybės kumulantų metodu.

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Autorės CV

1975-11-27 Gimiau Alytuje.

1983-1993 Alytaus 9-oji vidurinė mokykla (dabar "Jotvingių" gimnazija).

1993-1997 VGTU, Fundamentinių mokslų fakultetas,Inžinerinės informatikos mokslų bakalaurė.

1997-1999 VGTU, Fundamentinių mokslų fakultetas, Matematikos mokslų magistrė.

1999-2003 VGTU, Fundamentinių mokslų fakultetas, doktorant$\bar{u}$ros studijos.