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Developments in analytical wall shear stress modelling for water hammer phenomena

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ABSTRACT

New developments in analytical modelling of wall shear stresses during water hammer events are presented and discussed in this paper. These include further development of models based on the quasi-steady and unsteady hypothesis of hydraulic resistances. All the analyzed solutions have been improved so that they have extended ranges of applicability and the simplest possible mathematical form. Exemplary comparative studies were carried out, which showed their advantages and disadvantages. Explicit analytical formulas of wall shear stress were derived for the first time. They will be particularly useful at the stage of design and modernization of the pressurized piping systems (water supply, oil-hydraulics etc.) and enable effective analysis of leakages and other damages (ruptures, fatigue related failures) occurring in industrial pipeline systems. With their help, it will be easier to prevent the increased pulsating pressures occurring during water hammer events (protect against failures), and to verify numerical models used for years in simulations of transient pipe flows.

1. Introduction

In water supply, oil-hydraulics, district heating, etc., unsteady flows often occur. They are accompanied by sudden pressure pulsations that can damage the lines used for transporting the liquids. When we are dealing with accelerated, decelerated, reversed or pulsating flows and, in particular, when there are water hammer events (rapid closure of the valve, pump failure, water turbine emergency shutdown), the correct mathematical modelling is based on the form of the dynamic viscosity function.

For its quasi-steady form, complete analytical solutions are known. In the currently used numerical methods (NUM) based on the method of characteristics (MOC), the wall shear stress (WSS), which is responsible for the prediction of the friction forces associated with this type of flow, is modelled using the convolutional solution proposed by Zielke [1]. However, the classical method of determining the convolution from the local acceleration of the liquid and a weighting function is very time-consuming, therefore, analytical solutions would be very useful for designers of hydraulic systems, thanks to which it would be possible to accurately determine the

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Nomenclature

a, b, C	cross-section dependent constants
$a_{k,l}, b_{k,l}, A_{k,l}$	index and cross-section dependent functions
B	viscosity and radius dependent constant, [1/s]
c	pressure wave speed, [m/s]
D	pipe inner diameter, [m]
f	Darcy-Weisbach friction factor, [-]
$\mathbb{H}$	Heaviside unit step function, [-]
I_0, I_1, I_2	modified Bessel functions of the first kind, [-]
i	imaginary unit ($i = \sqrt{-1}$), [-]
J_0, J_1, J_2	Bessel functions of the first kind, [-]
k_n	index dependent function present in Rich solutions, [-]
L	pipe length, [m]
M	dynamic viscosity function, [-]
p	pressure, [Pa]
r	radial coordinate, [m]
R	pipe inner radius, [m]
Re	Reynolds number ($Re = \frac{vD}{\nu}$), [-]
s	Laplace variable, [1/s]
t	time, [s]
$\hat{t}$	dimensionless time ($\hat{t} = \frac{i}{R^2} t$), [-]
$T_{k,l}$	time and index dependent dimensionless function in Muto and Takahashi solution, [-]
u	time used in convolution integral, [s]
v	bulk velocity, [m/s]
$w(u)$	weighting function, [-]
Wh	water hammer number ($Wh = \frac{iL}{cR^2}$), [-]
x	distance, [m]
$\mathbb{Z}$	Joukowsky pressure rise ($\mathbb{Z} = \rho c v_0 $), [Pa]
η_j	roots of Bessel function $J_2(\eta_j) = 0$, [-]
$\lambda_{k,l}$	index and cross-sectional dependent function in Muto and Takahashi solution, [-]
μ	dynamic viscosity of liquid, [Pa·s]
$\dot{\nu}$	kinematic viscosity of liquid, [m ² /s]
ρ	liquid density, [kg/m ³]
τ	wall shear stress, [Pa]
φ	time and Wh number dependent function in Holmboe solution, [-]
ψ_n	index dependent function present in Rich quasi-steady solution, [-]
$\hat{\omega}_n$	k_n and Wh number dependent function in Mei-Jing solution. [-]

Indexes and abbreviations

F	final
cal	calculated
exp	experimental
res	reservoir
q	quasi-steady
u	unsteady
w	wall
0	initial
A-MJ	analytical Mei and Jing solution
A-MT	analytical Muto and Takahashi solution
A-QS	analytical quasi-steady Rich solution
CEF	complementary error function
CFD	computational fluid dynamics
CIB	convolution integral-based
IAB	instantaneous acceleration-based
MOC	method of characteristics
NUM	numerically obtained solution – with the help of MOC
RMSE	root mean square error
RPV	reservoir-pipe-valve system
WSS	wall shear stress

values of wall shear stress appearing in the conduits. The transient turbulent pipe flow requires the usage of different types of weighting functions in the convolution integral-based solutions (Reynolds number and roughness dependent), which were proposed by Vardy and Brown [2,3] or alternatively by Zarzycki [4]. The convolution integral can be solved in a more efficient way. Different efficient solutions of convolution integrals were published, firstly by Trikha [5], and next corrected in time by Kagawa et al. [6], Schohl [7] and Urbanowicz [8]. The efficient convolution integral solution forces the weighting function to be approximated by a finite (preferably small) number of exponentials terms. The other competing approach is based on Daily et al. [9] idea, that was in recent time improved by Brunone [10], Vítkovský et al. [11], Ramos et al. [12] and Cao et al. [13]. Since Brunone's paper [10] the so-called instantaneous acceleration (IA) group of models are mathematically defined as a sum of local and convective derivatives of bulk velocity multiplied by a damping coefficient. The water hammer flows have been recently investigated with the aid of CFD software. The recent findings in that field can be found in papers dedicated to 2-D and 3-D numerical approach [14–19].

In this work, in order to protect the pipes against excessive dynamic loads, simplified analytical formulas will be derived, with the help of which it will be possible to quickly determine the wall shear stress at any time from the moment of the forced impact. Using analytical solutions, designers of hydraulic, water supply, transmission, etc. systems will be able to protect the sensitive cross-sections of the pipes so as to prevent them from breaking and leading to leakages and blockages. With their help, it could be possible in the near future to online detect the anomalies occurring in monitored pipelines (leakages, rupture and other failures). Theoretical work on water hammer phenomena by neglecting frictional effects was initiated by Joukowski [20] and Allievi [21]. The Joukowski's formula for the pressure increase/decrease $\mathbb{Z}$ resulting from the valve closure is still used:

$$\mathbb{Z} = \rho c |\Delta v| \quad (1)$$

where ρ – liquid density, c – pressure wave speed, $\Delta v = v_0 - v_F$; v_0 – pre-transient (initial) flow velocity and v_F – final flow velocity (after valve closure).

Wood [22] took into account hydraulic resistance in a quasi-steady manner and developed few solutions in frequency domain. First analytical time domain water hammer solutions (for frictionless and quasi-steady case) were presented by Rich [23]. Unsteady frictional effects were properly demonstrated in the frequency domain by Iberall [24]. Many scientists in the 1960's tried to find a complete time domain analytical solution of Iberall's flow, no one has succeeded. The closest was Holmboe [25], who derived a simplified analytical formula for pressure at the valve cross-section for the first period of the water hammer event. At the same time, Zielke [26] developed a numerical solution for evaluation of laminar wall shear stress in pressurized pipelines. The unsteady part of the Zielke's model is defined by a convolution integral that includes an analytically determined weighting function and instantaneous local acceleration of the liquid. An interesting extension of the Holmboe solution was presented by Muto and Takahashi [27]. In this century, research on analytical solutions was undertaken by Sobey [28] and Jović [29]. Unfortunately, the solutions of both mentioned authors did not take into account unsteady wall shear stresses (like Rich only quasi-steady resistance was assumed). The solution originally used for blood flow modelling was developed by Mei and Jing [30,31]. Few limitations of this solution have been eliminated recently [32].

Due to the method discussed in the works by García García and Alvarino [33,34], it seems that there is a way for extending the analytical solutions defined so far for laminar flows, also for turbulent flows. The García-García and Alvarino model is based on the use of Pai's [35] approximation of the turbulent velocity profile. In this model the turbulent flow profile is composed as a sum of the Hagen–Poiseuille parabola (laminar profile) and a purely turbulent component. The main challenges during defining such a laminar-turbulent water hammer semi-analytical model will be the definition of the needed input data i.e. the mean pressure gradient during turbulent-laminar water hammer event and the weighted Reynolds shear stress gradient.

To sum up, despite numerous attempts, no one has managed to fully solve water hammer equations analytically even in the simplest reservoir-pipe-valve (RPV) type hydraulic system (Fig. 1). In most of the research works discussed in the introduction, the authors did not derive the equations for wall shear stress. Therefore, in this paper, novel formulas will be developed for the following three models: quasi-steady model of Rich [23] and two promising models that take into account the unsteady friction effects i.e. Muto and Takahashi [27], and Mei and Jing [30,31] models. Comparative studies of the developed simplified final formulas will be performed, based on the

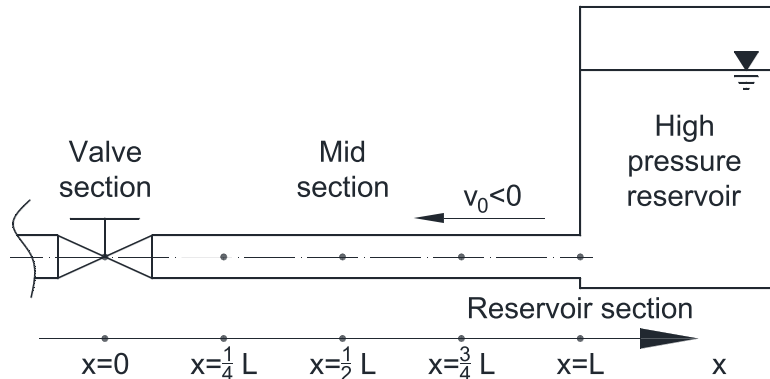


Fig. 1. Reservoir-pipe-valve system.

water hammer number (in the literature called also as fluid hammer number), which completely defines the transient laminar pipe flows [36]. Further guidance will be presented on how to develop a complete analytical model of water hammer.

A novel, faster method that is able to analyze transient pipe flows is nowadays especially sought. A group of authors promote and further develop frequency response methods [37,38]. Equation that in explicit manner represents solution in time domain will help to understand how the mechanism related to frequency-dependent friction works. Such knowledge will be particularly useful at the stage of designing systems of this type and at the stage of securing them, e.g. against leaks [39,40,41], shock responses [42] and other anomalies [43].

2. Analytical solutions

Water hammer in a horizontal pipe is described by the continuity and momentum equations as follows [44]:

$$\begin{cases} \frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial x} = 0 \\ \rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} + \frac{2}{R} \tau_w = 0 \end{cases} \quad (2)$$

where $p = p(x, t)$ – pressure, x – axial coordinate along the horizontal pipe, t – time, $v = v(x, t)$ – bulk velocity, R – pipe inner diameter and τ_w – wall shear stress.

The WSS can be calculated as a sum of quasi-steady and unsteady component [39,43]:

$$\tau_w = \tau_q + \tau_u \quad (3)$$

The quasi-steady component τ_q is calculated using the quasi-steady formula defined in the Ghidaoui et al. [45] review paper (Eq. 28 from Section 4.1 titled Quasi-Steady Wall Shear Models):

$$\tau_q = \frac{\rho f(x, t) v(x, t) |v(x, t)|}{8} \quad (4)$$

The friction factor in laminar flows is $f(x, t) = \frac{64}{Re(x, t)} = \frac{64\mu}{|v(x, t)| \cdot 2R}$, when introduced into Eq. (4) the quasi-steady WSS term is:

$$\tau_q = 4 \frac{\mu}{R} v(x, t) \quad (5)$$

The reader should be aware that in the most software (freeware and payable) used in water hammer engineering practice the quasi-steady WSS assumption is the only possible choice.

For unsteady term of the wall shear stress there are many approaches but two are important: convolutional integral-based (CIB) and instantaneous acceleration-based (IAB). Today both of these approaches are used for modeling purposes both for laminar and turbulent transient pipe flows. The CIB model for transient laminar pipe flow was developed by Zielke [11], who derived the unsteady part in the form:

$$\tau_u = 2 \frac{\mu}{R} \int_0^t w(t-u) \frac{\partial v(u)}{\partial t} du \quad (6)$$

where $w(t-u) = \sum_{j=1}^{\infty} e^{-\eta_j^2 \frac{t}{R^2}} \frac{1}{R^2}$ is a weighting function, having in time dependent exponential terms η_j which are consecutive roots of the Bessel function of the first kind and second order: $J_2(\eta_j) = 0$.

CIB model has been extended to turbulent flows [2,3,4]. In turbulent solution the weighting function in Eq. (6) is apart time, Reynolds number and pipe roughness dependent. The IAB models were developed initially by Daily et al. [9], while the modern forms of this model have been developed by Brunone [10], Vítkovský et al. [11] and Ramos et al. [12]. As this model is based on empirically defined friction coefficient, it will not be used in this work.

The case, when a quick (instantaneous) valve closure is assumed in a simple RVP system (Fig. 1), will be the topic of present investigations. The main objective of this work is to develop new explicit solutions for bulk velocity and wall shear stress, being step response solutions for flow rate input, whose simple mathematical form will enable their use in engineering practice at the design as well as the renovation stage of RVP systems.

2.1. Solution without consideration of friction losses

Let us first investigate a RVP system (Fig. 1) in which the influence of pipe wall friction is neglected i.e. $\tau_w = 0$. Rich [23] was the first who presented exact time-domain solution for pressure and flow velocity:

$$p(x, t) = p(L, 0) - 2\mathbb{Z} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{k_n c t}{L}\right) \sin\left(k_n \left(1 - \frac{x}{L}\right)\right)}{k_n \sin(k_n)} \quad (7)$$

$$v(x, t) = 2v_0 \sum_{n=1}^{\infty} \frac{\cos\left(\frac{k_n c t}{L}\right) \cos\left(k_n \left(1 - \frac{x}{L}\right)\right)}{k_n \sin(k_n)} \quad (8)$$

where $k_n = (n - 0.5)\pi$ and $p(L, 0)$ – reservoir pressure. In all equations from this point on we assume a full valve closure in the Joukowski formula i.e. v_F in Eq. (1) is equal to zero. Rich's solutions Eqs. (7) and (8) are Fourier square wave solutions. Similar solutions were presented relatively recently in Jović's book [29]. The difference between these solutions is that the Jović ones are based on Heaviside function.

2.2. Solution with consideration of quasi-steady friction

For a quasi-steady friction assumption Rich [23] derived analytical solutions for pressure and bulk velocity. Their unified final form is:

$$p(x, t) = p(L, 0) - 2Z \cdot Wh \cdot e^{-4i} \sum_{n=1}^{\infty} \frac{\sin(k_n(1 - \frac{x}{L})) [8\psi_n \cos(\psi_n \hat{t}) + (16 - \psi_n^2) \sin(\psi_n \hat{t})]}{\psi_n k_n^2 \sin(k_n)} \quad (9)$$

$$v(x, t) = 2v_0 e^{-4i} \sum_{n=1}^{\infty} \frac{\cos(k_n(1 - \frac{x}{L})) [\cos(\psi_n \hat{t}) + \frac{4\sin(\psi_n \hat{t})}{\psi_n}]}{k_n \sin(k_n)} \quad (10)$$

where $\psi_n = \sqrt{\left(\frac{k_n}{Wh}\right)^2 - 16}$, $\hat{t} = \frac{\hat{\nu}}{R^2} t$ – a dimensionless time and $\hat{\nu}$ – kinematic viscosity. The relation $Wh = \frac{\hat{\nu}}{cR^2}$ describes a very important dimensionless parameter called *water hammer number*. This important number can be represented as a function of two other very important and well-known dimensionless parameters: the *Reynolds number* for a pipe flow, i.e. $Re = (\nu D)/\hat{\nu}$ with ν the bulk flow velocity and D the pipe diameter, and the *Mach number*, i.e. $Ma = \nu/c$. This leads to the simple formula: $Wh = (Ma/Re) \cdot (2L/R)$ explaining more intuitively why $Wh \ll 1$ in many practical applications. This could also help to add interesting comments on the influence of turbulence, i.e. high Reynolds number flows correspond to a very low value of *water hammer number*, and on the influence of the fluid compressibility, i.e. high *Mach number* flows are associated with a high value of *water hammer number* (the authors would like to thank one of the reviewers to point out these practical issues and help to define a new useful form of Wh number). For more useful representations of *water hammer number* please refer to paper [36]. Recently taking the advantage of the similarity between electrodynamics and transient liquid pipe flow Jović [29] redeveloped the above solution. Similar solution for the quasi-steady friction assumption was derived by Sobey as well [28]. As in this paper the WSS is the main aim of the research for Rich's case it should be calculated with the help of Eq. (5), in which the bulk velocity defined by Eq. (10) is used:

$$\tau(x, t) = 4 \frac{\mu}{R} v(x, t) = 8 \frac{\mu v_0 e^{-4i}}{R} \sum_{n=1}^{\infty} \frac{\cos(k_n(1 - \frac{x}{L})) [\cos(\psi_n \hat{t}) + \frac{4\sin(\psi_n \hat{t})}{\psi_n}]}{k_n \sin(k_n)} \quad (11)$$

2.3. Solution with consideration of unsteady friction in laminar flow

If unsteady friction is taken into account, then the wall shear stress τ_w can be defined as a sum of quasi-steady and unsteady term: $\tau_w = \tau_q + \tau_u$. In the 60's of the previous century Brown [46], Holmboe [25], Zielke [26] and many others, based on Iberall's work [24], derived a Laplace domain solution of laminar water hammer and found an important dynamic viscosity function M :

$$M = \sqrt{\left(1 - \frac{2J_1(i\sqrt{\frac{s}{\nu}}R)}{(i\sqrt{\frac{s}{\nu}}R)J_0(i\sqrt{\frac{s}{\nu}}R)}\right)^{-1}} = \sqrt{\frac{-J_0(i\sqrt{\frac{s}{\nu}}R)}{J_2(i\sqrt{\frac{s}{\nu}}R)}} = \sqrt{\frac{I_0(\sqrt{\frac{s}{\nu}}R)}{I_2(\sqrt{\frac{s}{\nu}}R)}} \quad (12)$$

This function is a square root of the ratio of Bessel functions with arguments dependent on the Laplace variable s , kinematic viscosity $\hat{\nu}$ and pipe radius R . It plays the most important role as it affects the impedance and wave propagation operator. The last form (based on ratios of modified Bessel functions) presented on the r.h.s of Eq. (12) is more tractable for computation because it does not contain the imaginary unit $i = \sqrt{-1}$. An analysis of the course of the M function and an attempt to transform it to the time domain are presented in [47]. In all analytical formulas describing water hammer phenomena in the Laplace domain, this function and its square M^2 will play a very important role. A detailed discussion of its singularity points (infinite value) and transition from Laplace to the time domain of a function based on M^2 are presented in the appendix of Zielke's work [1]. If, however, the readers are interested in its graphical interpretation (Bode and polar plots) as well as in an extended discussion on approximation forms, they should consult the work by Ham [48] (pp. 18-19, Section 2.3 and Appendix pp. 146-151). For graphs of the magnitude and phase, please refer to Zhao et al. paper [49].

Due to its complicated structure Brown [46] derived an asymptotic expansion of it that is valid for large-modified Bessel function arguments $|\sqrt{\frac{s}{\nu}}R| > 3$:

$$M = 1 + \left(\frac{\dot{V}}{R^2 s}\right)^{1/2} + \left(\frac{\dot{V}}{R^2 s}\right) + \frac{7}{8} \left(\frac{\dot{V}}{R^2 s}\right)^{3/2} \quad (13)$$

Holmboe [25] used first three terms of this expansion (Eq. 13) to derive the following pressure solution:

$$p(0, t) = p(0, 0) + \mathbb{Z} \left[1 + \frac{2}{\sqrt{\pi}} \sqrt{\hat{t}} + \hat{t} + \frac{7}{6\sqrt{\pi}} \hat{t}^{3/2} - 2e^{-2Wh} \left\{ \operatorname{erfc}[\varphi^{-1/2}] + \frac{2\varphi^{1/2}}{Wh} \operatorname{ierfc}[\varphi^{-1/2}] + \frac{4\varphi}{Wh} \operatorname{i}^2 \operatorname{erfc}[\varphi^{-1/2}] + \frac{7\varphi^{3/2}}{Wh} \operatorname{i}^3 \operatorname{erfc}[\varphi^{-1/2}] \right\} \right] \quad (14)$$

where: $\varphi = Wh(\hat{t} - 2Wh)$ – water hammer number dependent function, $p(0, 0)$ – initial pressure at the valve section. Function $\operatorname{i}^n \operatorname{erfc}$ is n^{th} integral of the complementary error function CEF. The CEF is error function dependent: $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. The repeated integrals of the error function are given by [50]:

$$\operatorname{i}^n \operatorname{erfc}(x) = \int_x^\infty \operatorname{i}^{n-1} \operatorname{erfc}(u) du \text{ for } n = 1, 2, \dots \quad (15)$$

Exemplary: $\operatorname{i}^0 \operatorname{erfc}(x) = \operatorname{erfc}(x)$ and $\operatorname{i}^1 \operatorname{erfc}(x) = \pi^{-1/2} e^{-x^2} - x \cdot \operatorname{erfc}(x)$. The general recurrence formula for these repeated integrals is [50]:

$$2n \operatorname{i}^n \operatorname{erfc}(x) = \operatorname{i}^{n-2} \operatorname{erfc}(x) - 2x \operatorname{i}^{n-1} \operatorname{erfc}(x) \quad (16)$$

The Holmboe solution (Eq. 14) is valid only at an early stage of the water hammer event (first water hammer cycle: $\hat{t} < 4Wh$). Four first terms of the square bracket are responsible for the initial pressure rise (1/2 of water hammer period). Due to the limitations of this solution (valid only at the valve section and for the first cycle) Muto and Takahashi [27] for case of flow rate step response extended this solution for pressure in the time domain for all cross-sections along the RPV system and later cycles of water hammer. The final form of this solution is as follows:

$$p(x, t) = p(x, 0) + \mathbb{Z} \sum_{k=1}^2 \times \sum_{l=1}^{\infty} (-1)^{k+l-2} e^{-\lambda_{k,l} Wh} \left\{ (2 - \lambda_{k,l} Wh) \sqrt{\frac{T_{k,l} Wh}{\pi}} e^{-\frac{\lambda_{k,l}^2 Wh}{4T_{k,l}}} + \left[1 + Wh \left(T_{k,l} - \lambda_{k,l} + \frac{\lambda_{k,l}^2 Wh}{2} \right) \right] \operatorname{erfc} \left(\frac{\lambda_{k,l}}{2} \sqrt{\frac{Wh}{T_{k,l}}} \right) \right\} \mathbb{H} \left(T_{k,l} \frac{L}{c} \right) \quad (17)$$

where:

$\lambda_{k,l} = 2(k + l - 2) + \frac{x}{L}(-1)^{k+l-1}$; $T_{k,l} = t_{k,l}^c - \lambda_{k,l}$; $p(x, 0) = p(0, 0) + 8Wh \cdot \mathbb{Z} \cdot \frac{x}{L} p(0, 0)$ – is the valve pressure, and $\mathbb{H}$ – is a Heaviside step function.

The above solution (Eq. 17) corresponds to a typical water hammer situation that occurs just after the full instantaneous closure of the valve. Moreover, one can establish a relationship between the Holmboe and Rich parameters $\hat{t}$ and Wh , and Muto and Takahashi ones $T_{k,l}$ and $\lambda_{k,l}$ in the following form:

$$\frac{\hat{t}}{Wh} = T_{k,l} + \lambda_{k,l} \quad (18)$$

An improved solution that considers frequency dependent friction has been developed by Mei and Jing [30,31]. The authors investigated blood hammer in arteries due to sudden blockages. This model is valid for transient laminar flow. It should be noted that an error due line packing effect has been found in this model and had been corrected recently [32]. For the corrected equations provided by Jing please refer to Appendix A. The revised pressure, velocity and wall shear stress solutions, respectively, take the following form:

$$p(x, t) = p(L, 0) + 2\mathbb{Z} \sum_{n=1}^{\infty} \frac{\sin(k_n(1 - \frac{x}{L})) \left[\sin(\hat{t}\hat{\omega}_n) - \frac{8Wh}{k_n} \cos(\hat{t}\hat{\omega}_n) \right] e^{-i\sqrt{\frac{k_n}{2Wh}}}}{k_n \sin(k_n)} \quad (19)$$

where: $\hat{\omega}_n = \frac{k_n}{Wh} - \sqrt{\frac{k_n}{2Wh}}$.

$$v(x, t) = 2v_0 \sum_{n=1}^{\infty} \frac{\cos(k_n(1 - \frac{x}{L})) \left[\cos(\hat{t}\hat{\omega}_n) + \frac{8Wh}{k_n} \sin(\hat{t}\hat{\omega}_n) \right] e^{-i\sqrt{\frac{k_n}{2Wh}}}}{k_n \sin(k_n)} \quad (20)$$

$$\tau(x, t) = \frac{2\mu v_0}{R} \sum_{n=1}^{\infty} \frac{\sqrt{\frac{k_n}{2Wh}} \cos(k_n(1 - \frac{x}{L})) \left[\left(1 + \frac{8Wh}{k_n} \right) \cos(\hat{t}\hat{\omega}_n) + \left(\frac{8Wh}{k_n} - 1 \right) \sin(\hat{t}\hat{\omega}_n) \right] e^{-i\sqrt{\frac{k_n}{2Wh}}}}{k_n \sin(k_n)} \quad (21)$$

When one examines Eqs. (19) and (20) as well as (9) and (10), then it can be noticed that the developed form of the quasi-steady solution (pressure and velocity) and the one by Mei and Jing are similar. The main difference is the fact that in Rich's quasi-steady

solution the exponential damping term has a certain constant value (e^{-4i}) independent of the index of summation, while in Mei and Jing's unsteady solution the same expression depends on the index of summation and water hammer number ($e^{-i\sqrt{\frac{k_n}{2Wh}}}$). This means that the evolution of the phenomenon in Mei and Jing solutions is described by an essentially different function as in the Rich case. The time evolution in Mei and Jing case would not be self-similar and is, thus, a different type of evolution. Another important difference can be noticed in the arguments of the trigonometric functions found in the solution counters in square brackets. In Rich's quasi-steady solution, the excitation frequency is determined with the use of the function ψ_n being the root of the square of the ratio of the increasing circular function k_n to the water hammer constant minus a constant value of 16. However, in the Mei and Jing unsteady solution, the excitation frequency $\hat{\omega}_n$ is the difference from the ratio of the function k_n to the water hammer number Wh and the root of the same ratio multiplied by a constant equal to $1/\sqrt{2}$. It seems that a complete analytical solution (if it is possible to derive it one day) can also have this final form, and only the function contained in the argument will be different.

3. Recent developments in analytical velocity and wall shear stress solutions

This section presents and discusses recent novel solutions for bulk velocity and wall shear stress history [51,52] in order to complement the pressure history solutions originally developed by Holmboe [25], and Muto and Takahashi [27]. Let us start with the Holmboe's solution [25] in the Laplace-domain:

$$\tilde{p}(x, s) = \mathbb{Z} \frac{M}{s} \left[-e^{-\frac{x}{c}Ms} + e^{-\frac{(2L-x)}{c}Ms} + e^{-\frac{(2L+x)}{c}Ms} - e^{-\frac{(4L-x)}{c}Ms} - e^{-\frac{(4L+x)}{c}Ms} + \dots \right] + \frac{8v_0\dot{\nu}\rho}{sR^2}x \quad (22)$$

The spatial derivative of (Eq. 22) is:

$$\frac{\partial \tilde{p}(x, s)}{\partial x} = v_0\rho M^2 \left[-e^{-\frac{x}{c}Ms} - e^{-\frac{(2L-x)}{c}Ms} + e^{-\frac{(2L+x)}{c}Ms} + e^{-\frac{(4L-x)}{c}Ms} - e^{-\frac{(4L+x)}{c}Ms} - \dots \right] + \frac{8v_0\dot{\nu}\rho}{sR^2} \quad (23)$$

The bulk velocity equation in the Laplace-domain can be written in the following form:

$$\tilde{v}(x, s) = \left(\frac{1}{s\rho} \frac{\partial \tilde{p}}{\partial x} - \frac{8v_0\dot{\nu}}{s^2R^2} \right) \left(\frac{2J_1\left(i\sqrt{\frac{s}{\rho}}R\right)}{\left(i\sqrt{\frac{s}{\rho}}R\right)J_0\left(i\sqrt{\frac{s}{\rho}}R\right)} - 1 \right) - \frac{v_0}{s} = \left(\frac{8v_0\dot{\nu}}{s^2R^2} - \frac{1}{s\rho} \frac{\partial \tilde{p}}{\partial x} \right) \left(\frac{1}{M^2} \right) - \frac{v_0}{s} \quad (24)$$

Insertion of the spatial derivative (Eq. 23) into bulk velocity (Eq. 24) gives a different form of the bulk velocity in the Laplace-domain:

$$\tilde{v}(x, s) = -\frac{v_0}{s} \left[-e^{-\frac{x}{c}Ms} - e^{-\frac{(2L-x)}{c}Ms} + e^{-\frac{(2L+x)}{c}Ms} + e^{-\frac{(4L-x)}{c}Ms} - e^{-\frac{(4L+x)}{c}Ms} - \dots \right] - \frac{v_0}{s} \quad (25)$$

Taking now into account the first three terms of the Brown's asymptotic expansion of the dynamic viscosity function (Eq. 13), one gets:

$$M \cdot s = s + \sqrt{\frac{\dot{\nu}}{R^2}}\sqrt{s} + \frac{\dot{\nu}}{R^2} \quad (26)$$

After using the Eq. (26) in the exponential terms found in the square brackets of Eq. (25) they can be transformed back to time domain by using the inverse Laplace transform. The fourth term of Brown's asymptotic expansion Eq. (13) cannot be taken into account as the inverse Laplace transform of $e^{-a/s}$ (where a is positive constant) is undefined.

As an example, the first product term of the velocity solution (Eq. 25) will be evaluated. It has the following form:

$$FT_v = v_0 \frac{e^{-\frac{x}{c}\left(s + \sqrt{\frac{\dot{\nu}}{R^2}}\sqrt{s} + \frac{\dot{\nu}}{R^2}\right)}}{s} = v_0 \frac{e^{-as} \cdot e^{-b\sqrt{s}} \cdot e^{-\frac{a^2}{R^2}}}{s} = C \frac{e^{-as} \cdot e^{-b\sqrt{s}}}{s} \quad (27)$$

where: $a = \frac{x}{c}$; $b = \frac{x}{c} \sqrt{\frac{\dot{\nu}}{R^2}} = a \sqrt{\frac{\dot{\nu}}{R^2}}$ and $C = v_0 \cdot e^{-\frac{a^2}{R^2}} = \text{const.}$ Please note that all constants a , b and C used in Eq. (27) are constant in time but space dependent (x defines the analyzed cross-section).

By inspection one can see that in the remaining terms of the bulk velocity solution only the sign in the front of the subsequent exponential terms and the value of the a constants change (first term $a_1 = \frac{x}{c}$, second $a_2 = \frac{(2L-x)}{c}$, third $a_3 = \frac{(2L+x)}{c}$, fourth $a_4 = \frac{(4L-x)}{c}$ etc., and all have the positive values). The above guarantees that with the use of the second shift theorem:

$$\mathcal{L}^{-1} \left\{ \frac{e^{-as} \cdot e^{-b\sqrt{s}}}{s} \right\} = \mathbb{H}(t-a) \cdot \text{erfc} \left(\frac{b}{2\sqrt{t-a}} \right) \quad (28)$$

the novel time domain solution of the pipe flow bulk velocity reads as follows:

$$v(x, t) = v_0 \left(1 + \sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^l e^{-\lambda_{k,l} Wh} \operatorname{erfc} \left(\frac{\lambda_{k,l}}{2} \sqrt{\frac{Wh}{T_{k,l}}} \right) \mathbb{H} \left(T_{k,l} \frac{L}{c} \right) \right) \quad (29)$$

where: $T_{k,l} \frac{L}{c} = t - \lambda_{k,l} \frac{L}{c}$.

Finally, let us derive the wall shear stress (WSS) solution. The Holmboe's velocity profile distribution across the pipe area in the Laplace domain is [25]:

$$\tilde{u}(x, r, s) = \left(\frac{1}{s\rho} \frac{\partial \tilde{p}}{\partial x} - \frac{8v_0 \dot{\rho}}{s^2 R^2} \right) \left(\frac{J_0 \left(i \sqrt{\frac{s}{\nu}} r \right)}{J_0 \left(i \sqrt{\frac{s}{\nu}} R \right)} - 1 \right) - \frac{2v_0}{s} \left(1 - \frac{r^2}{R^2} \right) \quad (30)$$

Due to the fact that $\tilde{\tau}(x, s) = -\mu \frac{\partial \tilde{u}}{\partial r}$ where μ is dynamic viscosity, the derivative in respect to r from the above (Eq. 30) was first calculated:

$$\frac{\partial \tilde{u}}{\partial r} = \left(\frac{1}{s\rho} \frac{\partial \tilde{p}}{\partial x} - \frac{8v_0 \dot{\rho}}{s^2 R^2} \right) - i \frac{\sqrt{\frac{s}{\nu}} \cdot J_1 \left(i \sqrt{\frac{s}{\nu}} r \right)}{J_0 \left(i \sqrt{\frac{s}{\nu}} R \right)} \Bigg|_{r=R} + \frac{4v_0 r}{sR^2} \Bigg|_{r=R} \quad (31)$$

then

$$\tilde{\tau}(x, s) = -\mu \frac{\partial \tilde{u}}{\partial r} = -\frac{R}{2} \left(\frac{\partial \tilde{p}}{\partial x} - \frac{8v_0 \dot{\rho}}{sR^2} \right) \frac{2J_1 \left(i \sqrt{\frac{s}{\nu}} R \right)}{i \sqrt{\frac{s}{\nu}} R \cdot J_0 \left(i \sqrt{\frac{s}{\nu}} R \right)} - \frac{4\mu v_0}{sR} \quad (32)$$

and

$$\tilde{\tau}(x, s) = \frac{R}{2} \left(\frac{\partial \tilde{p}}{\partial x} - \frac{8v_0 \dot{\rho}}{sR^2} \right) \left(1 - \frac{2J_1 \left(i \sqrt{\frac{s}{\nu}} R \right)}{i \sqrt{\frac{s}{\nu}} R \cdot J_0 \left(i \sqrt{\frac{s}{\nu}} R \right)} - 1 \right) - \frac{4\mu v_0}{sR} = \frac{R}{2} \left(\frac{\partial \tilde{p}}{\partial x} - \frac{8v_0 \dot{\rho}}{sR^2} \right) \left(\frac{1}{M^2} - 1 \right) - \frac{4\mu v_0}{sR} \quad (33)$$

By inserting the derivative from (Eq. 23), one obtains the following equation for WSS in the Laplace domain:

$$\tilde{\tau}(x, s) = \frac{Rv_0 \rho}{2} \left[-e^{-\frac{s}{c} M \cdot s} - e^{-\frac{(2L-x)}{c} M \cdot s} + e^{-\frac{(2L+x)}{c} M \cdot s} + \dots \right] (1 - M^2) - \frac{4\mu v_0}{sR} \quad (34)$$

Similarly, as for the pressure and velocity solution in [48], the first three terms of Brown's high frequency expansion Eqs. (13) and (26) of the dynamic viscosity function are used to obtain the time domain solution of WSS:

$$\begin{aligned} \tau(x, t) = & \frac{v_0 \mu}{2R} \sum_{k=1}^2 \\ & \times \sum_{l=1}^{\infty} (-1)^l e^{-\lambda_{k,l} Wh} \left\{ \left(4 - \lambda_{k,l} Wh \right) \sqrt{\frac{T_{k,l} Wh}{\pi}} e^{\frac{\lambda_{k,l}^2 Wh}{4T_{k,l}}} + \left[3 + Wh \left(T_{k,l} + \frac{\lambda_{k,l}^2 Wh}{2} - 2\lambda_{k,l} \right) \right] \operatorname{erfc} \left(\frac{\lambda_{k,l}}{2} \sqrt{\frac{Wh}{T_{k,l}}} \right) + \frac{2e^{\frac{\lambda_{k,l}^2 Wh}{4T_{k,l}}}}{\sqrt{\pi \cdot Wh \cdot T_{k,l}}} \right\} \mathbb{H} \left(T_{k,l} \frac{L}{c} \right) \\ & - \frac{4v_0 \mu}{R} \end{aligned} \quad (35)$$

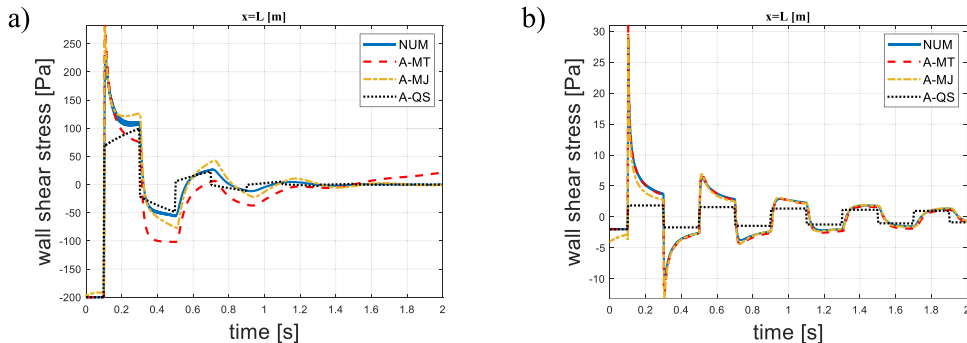


Fig. 2. Time history of WSS (at the reservoir cross-section) for: a) $Wh = 0.1$; b) $Wh = 0.01$ [52].

The most important results and conclusions from comparative studies [51,52] of the numerical solution (NUM) of the WSS obtained using the method of characteristics (MOC) and the analytical results, which were calculated using the models described by the following formulas: Muto and Takahashi A-MT (Eq. 35), Mei and Jing A-MJ (Eq. 21) and Rich A-QS (Eq. 11) will be recalled below. Selected tests carried out for two values of the water hammer number ($Wh = 0.1$ and $Wh = 0.01$) will be analyzed. The main conclusions from computational analysis presented in Fig. 2 indicate that:

- the main problem of the older version of analytical Muto and Takahashi model (A-MT) of WSS was the lack of fit, noticed especially for large values of Wh ($Wh > 0.05$ - see Fig. 2a). The A-MT model underestimates the shear stress at early stage of water hammer event, since $t = 0.2$ s, and at late stage (for $t \geq 1.4$ s) the calculated values of WSS start to grow in an uncontrolled way (failing to reach asymptotic solution). The smaller values of water hammer number (Fig. 2b) the less pronounced was this flaw in the earlier version of the WSS A-MT model. This model to be useful in all kinds of oil flows needs then a mandatory improvement;
- the quasi-steady Rich model (A-QS) of WSS failed to simulate even the early phase of the water hammer event (Fig. 2a for $t \leq 0.3$ s and Fig. 2b for $t \leq 1.1$ s);
- the unsteady model of WSS derived by Mei and Jing (A-MJ) exhibits problems with correct estimation of WSS values for initial phases of the water hammer even for small values of Wh number (Figs. 2a and 2b for times $t < 0.1$ s). Similar problem occurred in other test done for smaller values of Wh ($Wh < 0.01$).

Summarizing, the analysis carried out in the recent papers [51,52] indicated the need to improve analyzed analytical models, which take into account the frequency dependent friction, i.e. A-MT and A-MJ. In the remainder of this paper, we will focus on the improvement of the A-MT model, in particular the WSS solution.

3.1. Improved wall shear stress solution

As discussed above the A-MT model of WSS failed when the water hammer number Wh is larger than 0.01. To overcome this deficiency we propose an improved version of the A-MT model. To derive it we need to return to (Eq. 34) for WSS and use the complete inverse Laplace transform of M^2 function, which can be deducted from Ham's thesis [48] and Brereton-Jiang paper [53]:

$$\mathcal{L}^{-1} \left\{ M^2 = \frac{I_0 \left(\sqrt{\frac{\dot{\nu}}{\nu}} R \right)}{I_2 \left(\sqrt{\frac{\dot{\nu}}{\nu}} R \right)} \right\} = \delta(t) + 8 \frac{\dot{\nu}}{R^2} + 4 \frac{\dot{\nu}}{R^2} \sum_{j=1}^{\infty} \exp \left(-\eta_j^2 \frac{\dot{\nu}}{R^2} t \right) \quad (36)$$

The above time domain solution reveals that in the Laplace domain M^2 can be represented with a rational function based on zeros of Bessel function:

$$M^2 = 1 + 8 \frac{\dot{\nu}}{sR^2} + 4 \frac{\dot{\nu}}{R^2} \sum_{j=1}^{\infty} \frac{1}{\eta_j^2 \frac{\dot{\nu}}{R^2} + s} \quad (37)$$

When one applies the above (Eq. 37) form of M^2 function in WSS (Eq. 34) then the term $(1 - M^2)$ is expressed as: $-8 \frac{\dot{\nu}}{sR^2} - 4 \frac{\dot{\nu}}{R^2} \sum_{j=1}^{\infty} \frac{1}{\eta_j^2 \frac{\dot{\nu}}{R^2} + s}$. Assuming next for all powers in the exponential terms a high frequency behavior (Brown's asymptotic solution - (Eq. 26)):

$$e^{-a_1 M \cdot s} = e^{-a_1 \cdot \left(s + \sqrt{\frac{\dot{\nu}}{R^2}} \sqrt{s + \frac{\dot{\nu}}{R^2}} \right)} = e^{-a_1 s} \cdot e^{-a_1 \sqrt{\frac{\dot{\nu}}{R^2}} \sqrt{s}} \cdot e^{-a_1 \sqrt{\frac{\dot{\nu}}{R^2}}} \quad (38)$$

then one can write WSS:

$$\tilde{\tau}(x, s) = \frac{4\nu_0\mu}{R} \left(-\frac{1}{s} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\eta_j^2 \frac{\dot{\nu}}{R^2} + s} \right) \left(-e^{-a_1 M \cdot s} - e^{-a_2 M \cdot s} + e^{-a_3 M \cdot s} + \dots \right) - \frac{4\mu\nu_0}{sR} \quad (39)$$

$$\tilde{\tau}(x, s) = \frac{4\nu_0\mu}{R} \left(-\frac{1}{s} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\eta_j^2 \frac{\dot{\nu}}{R^2} + s} \right) \left[\sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^l e^{-a_{k,l} \left(s + \sqrt{\frac{\dot{\nu}}{R^2}} \sqrt{s + \frac{\dot{\nu}}{R^2}} \right)} \right] - \frac{4\mu\nu_0}{sR} \quad (40)$$

$$\tilde{\tau}(x, s) = \frac{4\nu_0\mu}{R} \left(-\frac{1}{s} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{\eta_j^2 \frac{\dot{\nu}}{R^2} + s} \right) \left[\sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^l A_{k,l} e^{-a_{k,l} s} e^{-b_{k,l} \sqrt{s}} \right] - \frac{4\mu\nu_0}{sR} \quad (41)$$

where:

$$A_{k,l} = e^{-a_{k,l}B}, b_{k,l} = a_{k,l}\sqrt{B} = a_{k,l}\sqrt{\frac{\dot{\nu}}{R^2}} \text{ and } B = \frac{\dot{\nu}}{R^2}.$$

The solution of (Eq. 41) can be split into two parts:

- a) quasi-steady (when the time and space dependent velocity defined by Eq. (25) and written in summation form is used in Eq. (5) defining the quasi-steady wall shear stress component)

$$\frac{4\mu}{R} \left[\underbrace{-\frac{v_0}{s} \sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^l A_{k,l} e^{-a_{k,l}s} e^{-b_{k,l}\sqrt{s}} - \frac{v_0}{s}}_{\tilde{v}(x,s)} \right] \quad (42)$$

- b) unsteady

$$-\frac{2v_0\mu}{R} \sum_{j=1}^{\infty} \frac{1}{\eta_j^2 \frac{\dot{\nu}}{R^2} + s} \cdot \left[\sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^l A_{k,l} e^{-a_{k,l}s} e^{-b_{k,l}\sqrt{s}} \right] \quad (43)$$

From Zielke's theoretical work [26] it is known that unsteady component of the wall shear stress can be written in the convolutional integral form - see Eq. (6). Similarly, here the product of two functions occurs. On Page 317 of Doetsch's book [54] it is stated that for $a \geq 0$:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathbb{H}(t-a) \quad (44)$$

Moreover, knowing that:

$$\mathcal{L}^{-1}\{e^{-b\sqrt{s}}\} = \frac{be^{-\frac{b^2}{4t}}}{2\sqrt{\pi t^3}} \quad (45)$$

the solution for the product of these two functions can be written:

$$\mathcal{L}^{-1}\{e^{-as} \cdot e^{-b\sqrt{s}}\} = \frac{be^{-\frac{b^2}{4(t-a)}}}{2\sqrt{\pi}(t-a)^{\frac{3}{2}}} \mathbb{H}(t-a) \quad (46)$$

The complete inverse Laplace transform of the unsteady WSS term is:

$$\tau_u = \frac{2\mu v_0}{R} \int_0^t \sum_{j=1}^{\infty} e^{-\eta_j^2 \frac{\dot{\nu}}{R^2}(u-t)} \sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^l A_{k,l} \frac{b_{k,l} e^{-\frac{b_{k,l}^2}{4(u-a_{k,l})}}}{2\sqrt{\pi}(u-a_{k,l})^{\frac{3}{2}}} \mathbb{H}(u-a_{k,l}) du \quad (47)$$

Solution of the above convolution integral (Eq. 47) and the inverse Laplace transform of the function described by the (Eq. 43) can be found in two well-known Laplace transform textbooks: Prudnikov et al. [55] on Page 54 and in Ederlyi's book [56] on Page 246. According to the formulas presented in these two books the unsteady WSS term has the following novel form in the time domain:

$$\tau_u(x, t) = v_0 \frac{\mu}{R} \sum_{j=1}^{\infty} \sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^{l-1} e^{-a_{k,l}B} \cdot e^{-\eta_j^2 B(t-a_{k,l})} \left[e^{-i\sqrt{\eta_j^2 B b_{k,l}^2}} \operatorname{erfc}\left(\frac{b_{k,l}}{2\sqrt{t-a_{k,l}}} - i\sqrt{\eta_j^2 B(t-a_{k,l})}\right) + e^{i\sqrt{\eta_j^2 B b_{k,l}^2}} \operatorname{erfc}\left(\frac{b_{k,l}}{2\sqrt{t-a_{k,l}}} + i\sqrt{\eta_j^2 B(t-a_{k,l})}\right) \right] \mathbb{H}(t-a_{k,l}) \quad (48)$$

where: i – unit imaginary number. Knowing that terms can be written as: $a_{k,l}B = \lambda_{k,l}Wh$; $t - a_{k,l} = T_{k,l}\frac{L}{c}$; $B(t-a_{k,l}) = T_{k,l}Wh$ then the above solution can be written in a form that clearly illustrates the important role of the water hammer number:

$$\tau_u(x, t) = v_0 \frac{\mu}{R} \sum_{j=1}^{\infty} \sum_{k=1}^2 \sum_{l=1}^{\infty} (-1)^{m-1} e^{-\lambda_{k,l}Wh} e^{-\eta_j^2 Wh T_{k,l}} \left[e^{-i\eta_j \lambda_{k,l} Wh} \operatorname{erfc}\left(\frac{\lambda_{k,l} Wh}{2\sqrt{T_{k,l} Wh}} - i \cdot \eta_j \sqrt{T_{k,l} Wh}\right) + e^{i\eta_j \lambda_{k,l} Wh} \operatorname{erfc}\left(\frac{\lambda_{k,l} Wh}{2\sqrt{T_{k,l} Wh}} + i \cdot \eta_j \sqrt{T_{k,l} Wh}\right) \right] \mathbb{H}\left(T_{k,l} \frac{L}{c}\right) \quad (49)$$

Wolfram Mathematica program was used to perform simulations.

3.2. Wall shear stress at the valve section

It is also possible to derive a novel complete analytical solution of the WSS at the valve section. When one examines the (Eq. 34) and inserts $x = 0$ (valve section location), then exponential terms in square bracket (except for the first one, which will take the value -1) reduce themselves:

$$\tilde{\tau}(0, s) = \frac{Rv_0\rho}{2} \left[-e^{-\frac{0}{c}Ms} - e^{-\frac{(2L)}{c}Ms} + e^{-\frac{(2L)}{c}Ms} + \dots \right] (1 - M^2) - \frac{4\mu v_0}{sR} = -\frac{Rv_0\rho}{2} (1 - M^2) - \frac{4\mu v_0}{sR} \quad (50)$$

Next if the function M^2 of (Eq. 37) is inserted:

$$\tilde{\tau}(0, s) = -\frac{Rv_0\rho}{2} \left(1 - 1 - \frac{8}{s} \frac{\dot{U}}{R^2} - 4 \frac{\dot{U}}{R^2} \sum_{j=1}^{\infty} \frac{1}{s + \frac{\dot{U}}{R^2} \eta_j^2} \right) - \frac{4\mu v_0}{sR} = \frac{2v_0\mu}{R} \sum_{j=1}^{\infty} \frac{1}{s + \frac{\dot{U}}{R^2} \eta_j^2} \quad (51)$$

The inverse Laplace transform of the above WSS operator at the valve section is:

$$\tau(0, t) = \frac{2v_0\mu}{R} \sum_{j=1}^{\infty} \exp\left(-\eta_j^2 \frac{\dot{U}}{R^2} t\right) \quad (52)$$

The Eq. (52) is the complete analytical solution for WSS at the valve section which is closed instantaneously. With the use of it (Eq. 52) one can check the difference between this complete analytical solution with the proposed revised solution (Eq. 49) for WSS (valid for all x from range $0 \leq x \leq L$) and numerical solution.

4. Experimental and numerical verification of novel solutions

The comparison of the results calculated by analytical solutions will be made with the results of measurements taken by Bergant et al. [57] and Adamkowski and Lewandowski [58] (comparison of pressure histories) and with the results of numerical tests (time histories of WSS and velocities). The main emphasis of this investigation will be on analysis of dynamic histories of WSS and velocities at the selected cross-sections of the tested RPV system, as these solutions have not been analysed yet [32]. In numerical studies (NUM), the method of characteristics (MOC) was used to solve the system of partial differential equations describing water hammer (Eq. 2). MOC is used as it perfectly interprets the physical essence of the transient flow phenomenon. It is characterized by fast convergence and high accuracy, and it enables relatively simple incorporation of complex boundary conditions. Most details about the MOC code used in our simulations will not be repeated here as they were discussed in [8,59]. The friction model used in NUM calculation is CIB. The unsteady part of wall shear stress (Eq. 6) is solved by using an efficient method that approximates the complete weighting function to the following form:

$$w(t) = \sum_{j=1}^{\infty} e^{-\eta_j^2 \frac{\dot{U}}{R^2} t} \approx \sum_{j=1}^3 a_j e^{-b_j t} \quad (53)$$

The constant coefficients a_j and b_j are numerical time step dependent, which means that they vary depending on the simulated case. More details about their calculation and form of efficient solution can be found in [59].

In this work the focus is put on the effect of frequency dependent friction that occurs in transient laminar flows in elastic pipes (metal). We hope that the derived new formulas (solutions) will allow for a detailed understanding of the nature of the pipe wall (skin) friction losses. Novel solutions discussed in this paper would be helpful to find a solution for turbulent flow too. An interesting model based on the Theory of Underlying Laminar Flow (TULF), which has been proposed in recent years by García García and Alvarino [33, 34], would be a good starting point. The solutions presented in this paper are written in a unified form (the quasi-steady and all frequency-dependent solutions are water hammer number dependent). The complete history of the unifying process of the original functions can be followed in earlier papers by authors [32,51,60].

In Section 4.1 the overall correctness of the analytical models will be examined by comparison with the experimental results. In the next Section 4.2 all the novel analytical solution results will be compared with the numerical results over a broad range of water hammer number. The results will be analyzed at three characteristic cross-sections of the analyzed RPV system: at the valve cross-section ($x = 0$); at the midpoint ($x = 0.5L$) and at the reservoir section ($x = L$) – see Fig. 1.

4.1. Comparison with experimental results

In liquid-filled pipes the unsteady pressure measurement is traditionally the most used measurement technique. Unsteady pressure is the only measured quantity that can be recorded and processed with high accuracy, availability and low costs. This is not the true for the measurement of unsteady flow velocity and wall shear stress. That's why one can find several publications with experimental transient pressure results. A number of them present water hammer in a single horizontal pipeline. For our comparisons two distinct experimental runs have been selected. The first one was conducted by Bergant et al. [57] and the second one by Adamkowski and Lewandowski [58]. The basic data describing the test stand flow conditions which are needed to perform comparisons are collected in Table 1. Transient event is triggered by sudden closure of the downstream end valve in both RPV systems.

Typically pressure pulsations are recorded at the cross section located close to the downstream end valve. It is this section at which the pressure rise lasts the longest time during valve closure event. In the remaining cross-sections, the pressure of a nearly equal high level is kept for a shorter time. The comparisons carried out in [32] investigated conditions at the valve section. In order to test the presented analytical solutions for predicting transient pressure at different locations, simulations in this paper are performed at the midpoint of the pipeline.

The following results at the midpoint of the two studied RPV systems are presented and compared in Fig. 3: numerical calculations (NUM) using a code based on MOC; measurements (EXP); analytical Mei and Jing model – (Eq. 19) (A-MJ), analytical Muto and Takahashi model – (Eq. 17) (A-MT), and quasi-steady Rich model simulations – (Eq. 9) (A-QS).

The performed comparative simulations of pressure time histories (Fig. 3) at the midpoint cross-section of the pipe (related to two exemplary experimental results) showed that:

- (i) for lower values of the Wh number ($Wh < 0.01$), there is a good fit between calculated histories of A-MJ and A-MT models. However, the A-MT model performs a little bit better than A-MJ model. The later slightly underestimates overpressures and overestimates under pressures;
- (ii) the numerical solution NUM and the analyzed analytical models that take into account the unsteady friction (A-MJ and A-MT) match the experimental results well in the sense of compliance of dissipation and dispersion;
- (iii) the quasi-steady friction model A-QS fails to calculate properly the time histories of pressures obtained experimentally. Studies have shown that the A-QS approach can only be used when the value of the water hammer number is very small $Wh \leq 1 \cdot 10^{-5}$ or very large $Wh \geq 0.5$. When Wh has a small value, after a longer time, a discrepancy in the results is visible, thus this model simulates the pressures with sufficient consistency only in the initial phase of the water hammer event. As can be seen from Figs. 3a and 3b, the pressure time histories modeled with this model are very inaccurate. Pressure changes are predicted as straight lines; thus, they do not properly reflect the experimentally observed waveform with rounding. Generally, the A-QS model fails to model properly damping (attenuation of pressure fluctuation amplitudes) and dispersion (time delay or phase-shift of pressure waves);
- (iv) for in-depth analysis, it is desirable to study not only the starting period of the water hammer event (early phase), but also amplitudes observed in the later phase of the event. The phase of transient run, when the differences between the simulated and experimental results can be significant, is dependent on Wh number;
- (v) the method of characteristics numerical model and analytical solutions (considering frequency-dependent friction) are highly effective. These models are known from the literature, however, the analytical formulas have been extended and simplified.

Quantitative analysis in this subsection and in the original reference paper [27] shows discrepancies for large values of $Wh > 0.1$. Readers interested in this problem are referred to the work [32].

Qualitative analysis of the results presented graphically in Fig. 3 is performed with the use of RMSE:

$$RMSE = \sqrt{\frac{\sum_{j=1}^N (p_{j,cal} - p_{j,exp})^2}{N}} \quad (54)$$

where: $p_{j,exp}$ – pressure values measured experimentally; $p_{j,cal}$ – pressures calculated numerically or analytically, N – number of analyzed data points.

RMSE measures the differences between numerically calculated values and those obtained from experimental tests. The results are collected in Table 2.

The obtained qualitative results (Table 2) indicate that the numerical (NUM) and analytical models that take into account the unsteady frictional losses (A-MJ and A-MT) perform closely to the experimental results, while the quasi-steady analytical solution A-QS performs poorly. The smaller values of RMSE for all analyzed models in the case of Bergant et al. is the effect of smaller pressure pulsations recorded experimentally (especially in first two amplitudes). The experimental data used in these comparisons as seen in Fig. 3 are raw measurements, the filtration of it would help to get smaller RMSE results. Although the best fit among the analyzed models was obtained using the numerical model (NUM), it should be remembered that analytical models have an advantage over the numerical method in that they can be used to model a selected time interval of unsteady flow (while in NUM the complete pressure histories need to be calculated since the beginning of transients).

4.2. Performance of improved analytical model

Due to the lack of experimental data of WSS and bulk velocity, the simulations performed herein are compared to the numerical tests carried out with the help of the MOC model. Comparative simulations of wall shear stress (WSS), bulk velocity (for simplicity in

Table 1
Basic data for two experimental runs in RPV systems.

System	L [m]	R [m]	$\dot{v}$ [m ² /s]	c [m/s]	p_{res} [Pa]	v_0 [m/s]	ρ [kg/m ³]	Wh [-]	Re [-]
Bergant et al. [57]	37.23	0.01105	$1.13 \cdot 10^{-6}$	1302	$4.15 \cdot 10^5$	0.1	999.1	$2.65 \cdot 10^{-4}$	1956
Adamkowski & Lewandowski [58]	98.11	0.008	$9.5 \cdot 10^{-7}$	1308	$1.265 \cdot 10^6$	0.066	997.65	0.0011	1112

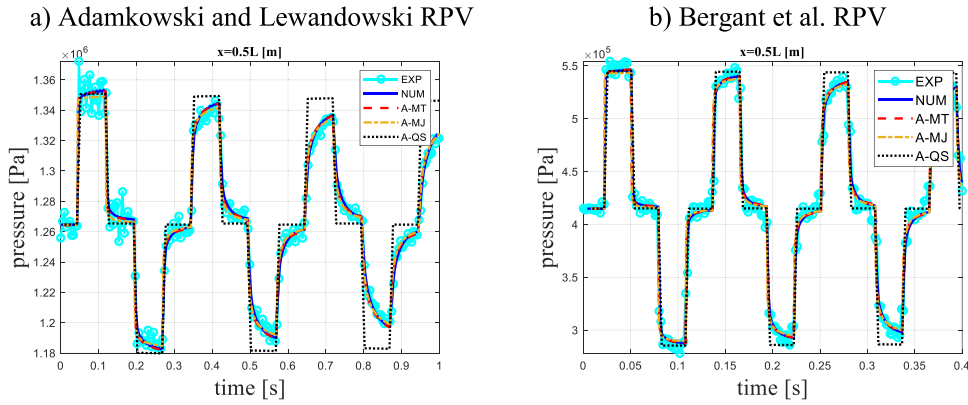


Fig. 3. Comparison of pressure time histories at the midpoint of the two selected RPV systems.

Table 2
RMSE analysis results.

Model	Analyzed cases:	
	Adamkowski and Lewandowski	Bergant et al.
Numerical	$3.98 \cdot 10^3$ [Pa]	$1.12 \cdot 10^3$ [Pa]
A-MT	$5.25 \cdot 10^3$ [Pa]	$1.79 \cdot 10^3$ [Pa]
A-MJ	$5.44 \cdot 10^3$ [Pa]	$1.93 \cdot 10^3$ [Pa]
A-QS (quasi-steady)	$2.23 \cdot 10^4$ [Pa]	$1.28 \cdot 10^4$ [Pa]

this section called only velocity) and pressure time histories were performed at all three key conduit cross-sections and for three selected water hammer numbers Wh : 0.001; 0.01 and 0.1. The Wh values have been controlled by the liquid viscosity. The selected values are important because they define the type of liquid and consequently flow conditions: $Wh = 0.001$ (water), $Wh = 0.01$ (water-oil emulsion), and $Wh = 0.1$ (oil). The data necessary for the simulation of the analyzed RPV system are summarized in Table 3.

Calculated results are presented in Figs. 4 to 12: (i) simulations for WSS time histories for three simulated Wh cases in Figs. 4 to 6 (results for all three cross-sections $x = 0$; $x = 0.5L$ and $x = L$ are analyzed in charts); (ii) for velocity changes in Figs. 7 to 9 (results for $x = 0.5L$ and $x = L$ are presented; for $x = 0$ the velocity for $t > 0$ is set to be zero) and finally, (iii) for pressure time histories in Figs. 10 to 12 (results for $x = 0$ and $x = 0.5L$ are presented; for $x = L$ the pressure for any time is set to have a constant value), respectively.

Comparisons of the results obtained for the WSS time histories show that:

- (i) the A-MJ model (Eq. 21) has difficulties in correctly modeling the initial phase of the water hammer (Figs. 4b, d and Figs. 5b, d). These problems are revealed in the form of inaccurate calculations of the WSS in the initial time until the primary wave of the increased pressure arrives at the analyzed section (at the midpoint $x = 50$ m, $t < 0.05$ s - Fig. 4d and Fig. 5d; while for the section at the reservoir $x = 100$ m, $t < 0.1$ s - Fig. 4b and Fig. 5b). Before the arrival of the initial pressure wave, there is still steady motion at these sections and the values of analyzed parameters should be the same as in the pre-transient (initial) state. From the same results (runs for which $Wh \leq 0.01$), during the first two periods of the water hammer event for runs with $Wh \leq 0.01$ this model underestimates the WSS at the reservoir cross-section and at the midpoint one. In tests where $Wh = 0.1$ (Fig. 6) we see that alternately the WSS is overestimated and underestimated at the midpoint section and at the reservoir (Figs. 6a, b, c, d), although the deviations from the numerical results are still acceptable from a practical point of view. It happens, as can be seen from Fig. 4d, that with its help it is possible to significantly overestimate the peak WSS value ($t = 0.05$ s). As it is a simplified model, it does not correctly model the WSS values at the valve cross section (Figs. 4e, f; Figs. 5e, f and Figs. 6e, f). At this cross-section, the results determined with its help are zero, and in fact, a gradual stress relaxation should be observed, which is

Table 3
Analyzed cases data.

Wh	0.001	0.01	0.1
$\dot{\nu}$ [m ² /s]	$1 \cdot 10^{-6}$	$1 \cdot 10^{-5}$	$1 \cdot 10^{-4}$
ν_0 [m/s]	-0.05	-0.5	-5

Other: $L = 100$ m; $R = 0.01$ m;
 $c = 1000$ m/s, $\rho = 1000$ kg/m³,
 $p_{res} = 1 \cdot 10^8$ Pa; $Re = 1000$

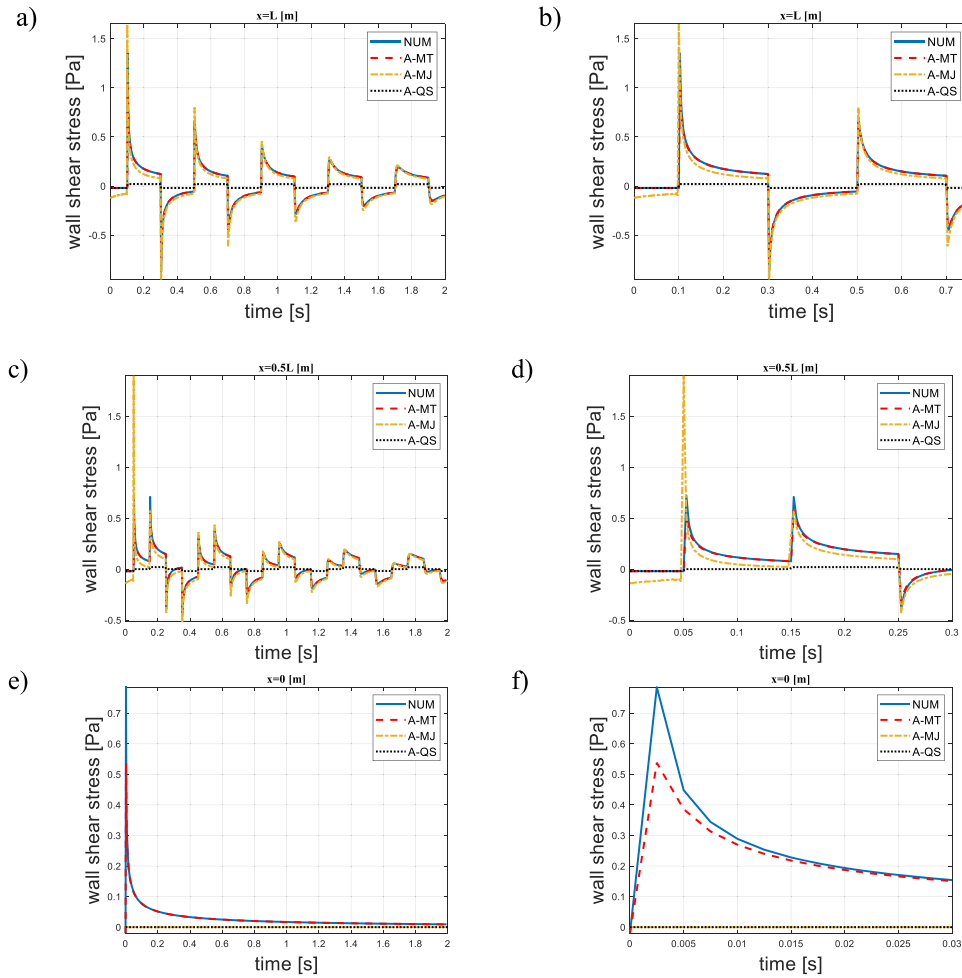


Fig. 4. WSS time histories for a typical water flow ($Wh = 0.001$).

confirmed by the calculations made with the A-MT model and numerically determined NUM. Despite this deficiency, from the third amplitude on (Figs. 4a, c; Figs. 5a, c) and in the entire run for $Wh = 0.1$ (Figs. 6a, c) it simulates the WSS reasonably well and does not deviate from the A-MT analytical model (except for the cross-section at the valve);

- (ii) the improved A-MT model (which was presented for the first time in this paper – Eq. 49) predicted very well time histories of WSS at the reservoir and midpoint cross-sections of the analyzed RPV system (Figs. 4a, b, c, d and Figs. 5a, b, c, d) in all tests in which a relatively small value of the water hammer number was maintained, i.e. $Wh \leq 0.05$. Only slight differences in peak values can be noticed when compared to MOC results. In the case when $Wh = 0.1$ one can see deviations from the numerical time histories of WSS (Figs. 6a, b, c, d). These deviations are mainly due to the fact that the quasi-steady values of the WSS calculated with the help of the velocity solution (Eq. 29) starts to be predicted with a further increase of Wh (over the value 0.1) with an increasing error. This error is the result of an inaccurate asymptotic value of the velocity model. Additional tests (not presented in graphical form herein) showed that the value simulated by the formula (Eq. 29) begins to significantly differ from the value obtained by means of the numerical method for $Wh > 0.1$. The above problem indicates the need for further modification of the velocity solution so that, similarly to the A-MJ model, it tends to the correct asymptotic value. From the analysis of the WSS results obtained at the valve cross-section (Figs. 4e, f; Figs. 5e, f and Figs. 6e, f), some significant model differences between the numerical results (NUM) and the analytical results (A-MT) can be observed. The numerically simulated WSS peak is much larger than one that is obtained from the analytical model (Fig. 4f; Fig. 5f and Fig. 6f). Due to the fact that the results obtained with the proposed (Eq. 49) are very close to the analytical results obtained from the complete solution of WSS (Eq. 52) at the valve cross-section ($x = 0$) they indicate that an error is made in the numerical approach. However, further research is needed to find out in detail what is the reason for the observed differences. Probably the source of the error will be the linearization in the numerical solution of the convolutional integral of velocity changes during a single numerical time step;
- (iii) the A-QS model (Eq. 11) developed by Rich, for which the WSS time histories are obtained for $Wh \leq 0.05$ is characterized by poor compliance with the actual WSS time histories calculated with the help of numerical MOC model. With an increase of $Wh > 0.2$ A-QS model begins to approach the numerical results. In the comparisons for $Wh = 0.1$, in particular in Figs. 6a, b, c, d it can

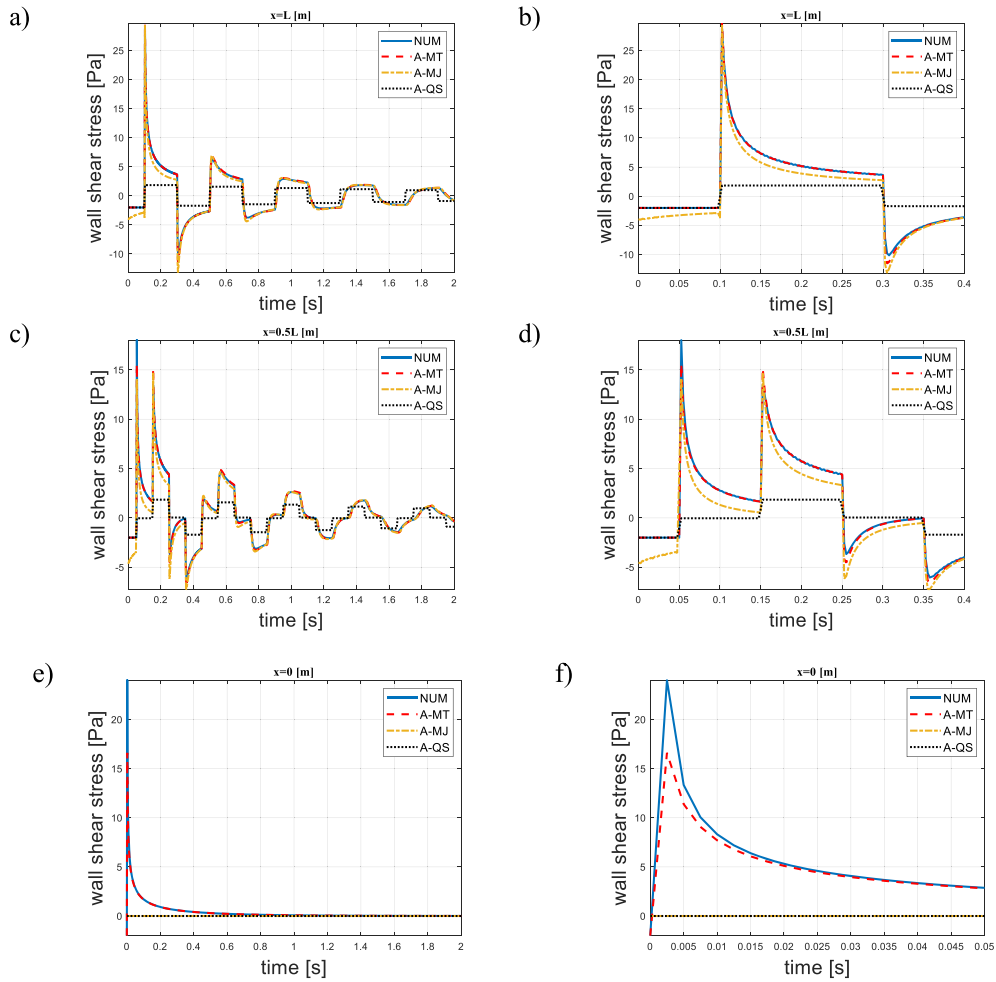


Fig. 5. WSS time histories for a typical water-oil mixture flow ($Wh = 0.01$).

be seen that from the second amplitude on the peaks modelled with A-QS model are close to the numerical ones. The largest mismatch occurs in the initial phase, at the first amplitude, where there is a WSS jump followed by its relaxation (Figs. 6a, b; $0.1 \leq t \leq 0.3$). A similar situation is also observed at the midpoint section (Figs. 6c, d; $0.05 \leq t \leq 0.25$). Generally, this model does not model damping and dispersion. The phase shift visible in the results obtained with the help of frequency dependent models are not modeled with the help of frequency dependent models correctly. The calculated period of water hammer is always $t = 4L/c$. This means that the model does not take into account physical wave dispersion effects (effects of radial diffusion);

- (iv) in comparison to the results obtained with the aid of the original version of the A-MT WSS model defined by (Eq. 35), which was analyzed in conference papers [48,49], a significant improvement and extension of the applicability range of the modified analytical model of WSS (Eq. 49) were obtained. In the range $0.2 \leq Wh \leq 1$, however, it is still necessary to further modify this model, in particular by improving the solution for the velocity $v(x, t)$ of the flow (Eq. 29) which has a significant impact on the modelled WSS time histories.

The conducted comparative studies of the velocity time histories show that both models that take into account unsteady friction, i.e. A-MT (Eq. 29) and A-MJ (Eq. 20), perform consistently for the cases with small values of the water hammer number ($Wh \leq 0.001$) (Fig. 7). Qualitative analysis for these Wh values shows no problems with matching the time histories calculated by analytical models with those obtained by numerical simulations. The simplified quasi-steady A-QS model (Eq. 10), apart from a good phase adjustment, was not able to mimic the dynamics of changes. The A-MT and A-MJ models are also suitable for modeling the time histories when $Wh = 0.01$ (Fig. 8), however, here the first amplitude simulated with the help of the A-MJ model differs when compared to the numerical results (NUM). Also in this case, the A-QS model did not perform adequately in terms of the velocity time histories. Figure 9 shows the time histories of velocity determined for a relatively large Wh number i.e. $Wh = 0.1$. From these comparisons, it is evident that the A-MJ model incorrectly estimates the time histories of velocity in the initial period of the impact, including at the peak of the first amplitude, where it overestimates the velocity value by more than 1 m/s. In the final phase of the run, which is very important, the results obtained with the help of this model tend to the correct asymptotic value.

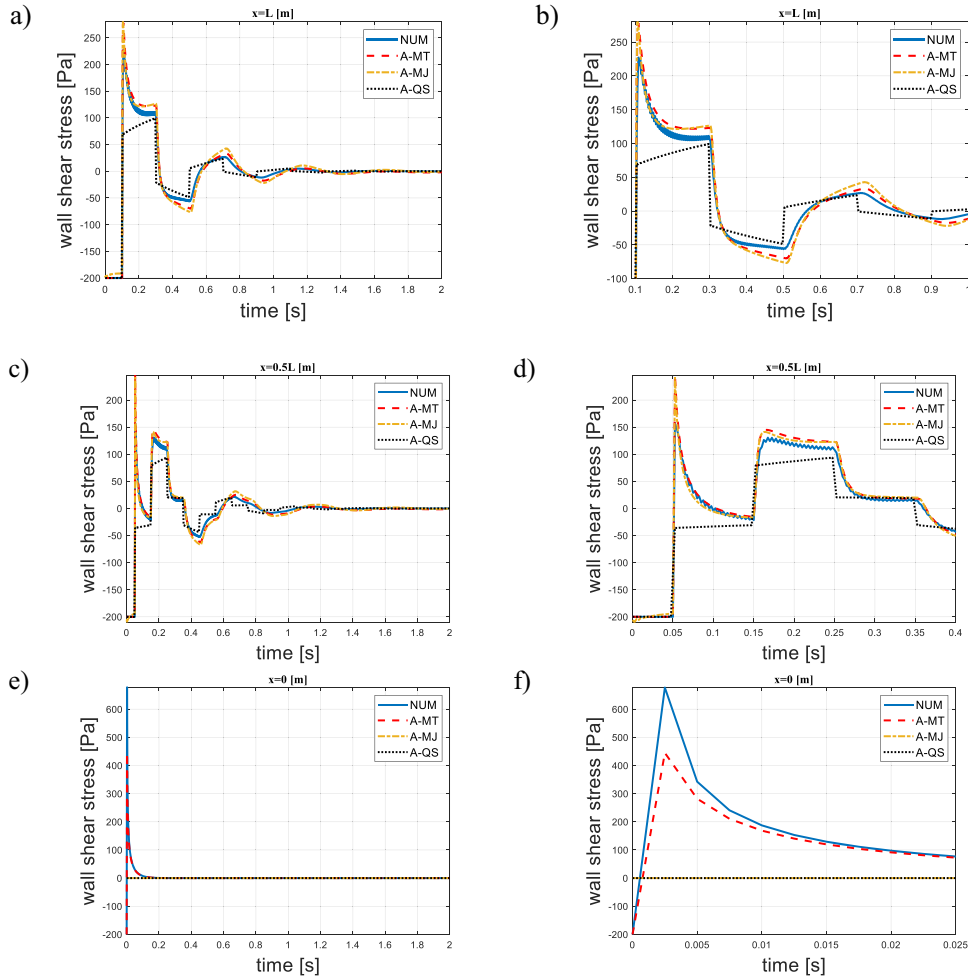


Fig. 6. WSS time histories for a typical oil flow ($Wh = 0.1$).

The A-MT model slightly overestimates the time histories of velocity from the first amplitude, but generally still satisfies the numerical results. Unfortunately, further numerical studies (not showed graphically in this paper) for larger values of $Wh \geq 0.2$ show that the asymptotic velocity values simulated with this model are inaccurate. This finding indicates the need for further improvement of this model (A-MT) in the time domain, so that the asymptotic value is correctly modelled.

Finally, comparisons of pressure time histories are presented and discussed. The simulations were performed with the help of solutions defined by (Eq. 9) for A-QS, (Eq. 17) for A-MT and (Eq. 19) for A-MJ model, respectively. The computed results for $Wh = 0.001$ (Fig. 10) showed that the A-MJ model simulates the peaks and valleys of the successive initial pressure amplitudes with a slight but visible inaccuracy - this applies to cross sections at the valve (Figs. 10c, d) and the midpoint (Figs. 10a, b). The A-MT model simulates the time histories of pressure changes for the same value of the Wh number with high consistency (it also works well for larger times, not graphically illustrated in the figures discussed, correctly simulating further damping of the time histories of pressure). In the tests for $Wh = 0.01$ (Fig. 11), in an early phase of the water hammer event (the first period), the pressure estimation error with the A-MJ model becomes more and more noticeable. However, with each successive period, the model differences fade away and both the A-MJ and A-MT models simulate the remaining stages of the pressure histories with good consistency. For this value of the Wh number, even for relatively larger times, there is a good match of pressure histories when calculations were made with the A-MT model. However, the situation begins to change rapidly with a further increase of the water hammer number: $Wh \geq 0.1$ (Fig. 12).

The quasi-steady history of A-QS begins to approach the numerical history when it comes to estimation of the maximum pressure values at successive amplitudes (only the first three are well noticeable). However, there are still noticeable sudden jumps in pressure values that do not occur in numerical calculations, and a phase error. The A-MT model underestimates pressure values from the first detected pressure drop after the first amplitude. Additionally, from the time of about 1.3 to 1.4 s obtained by this model, the pressure value starts to increase linearly (Figs. 12a, c) - this is not justified, so this model has a limited range of applicability. The A-MJ model for the value of $Wh = 0.1$ overestimates the pressure at successive initial amplitudes and worsens the phase convergence, however, it always tends to the correct final (asymptotic) value.

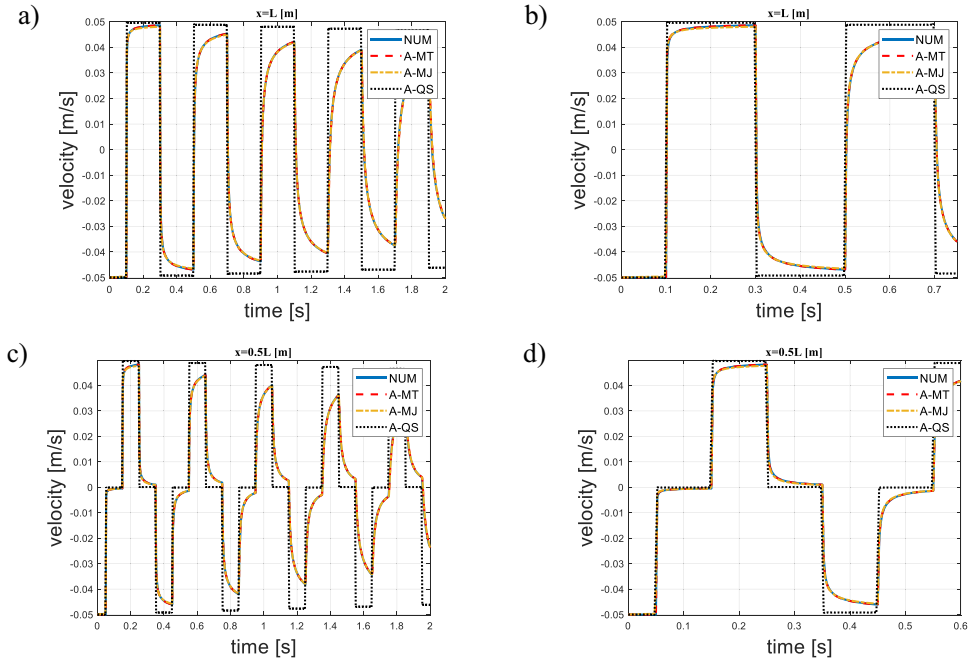


Fig. 7. Velocity time histories for a typical water flow ($Wh = 0.001$).

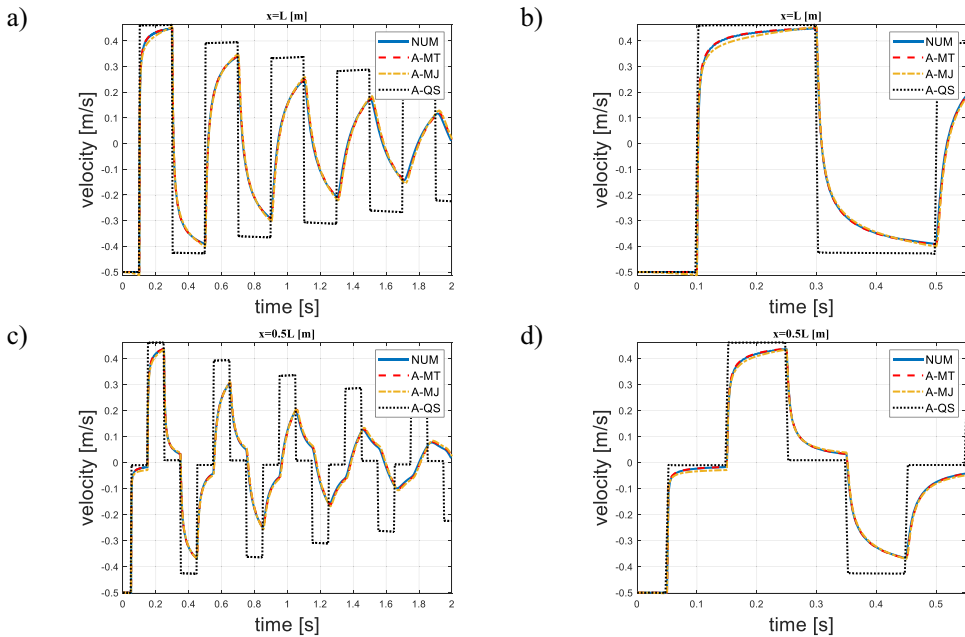


Fig. 8. Velocity time histories for a typical water-oil mixture flow ($Wh = 0.01$).

5. Conclusions

The paper presents new forms of analytical water hammer solutions in the time domain, which were developed from the analytical solutions in the Laplace domain. Solutions have been developed for frictionless reservoir-pipe valve (RPV) system, RPV system with consideration of quasi-steady friction and finally, RPV system considering laminar flow-based frequency-dependent friction. Summing up, the presented analytical solution of the wall shear stress (WSS) is characterized by a large advantage over the previously used solution. Large errors of the older solution have been eliminated i.e. uncontrolled increase of the shear stress in the later phase of the water hammer event and increase of the convergence error when $Wh \geq 0.01$. The simulation tests carried out showed the effectiveness not only of the novel analytical model of WSS but also of bulk velocity and pressure models. Presented research revealed that extended

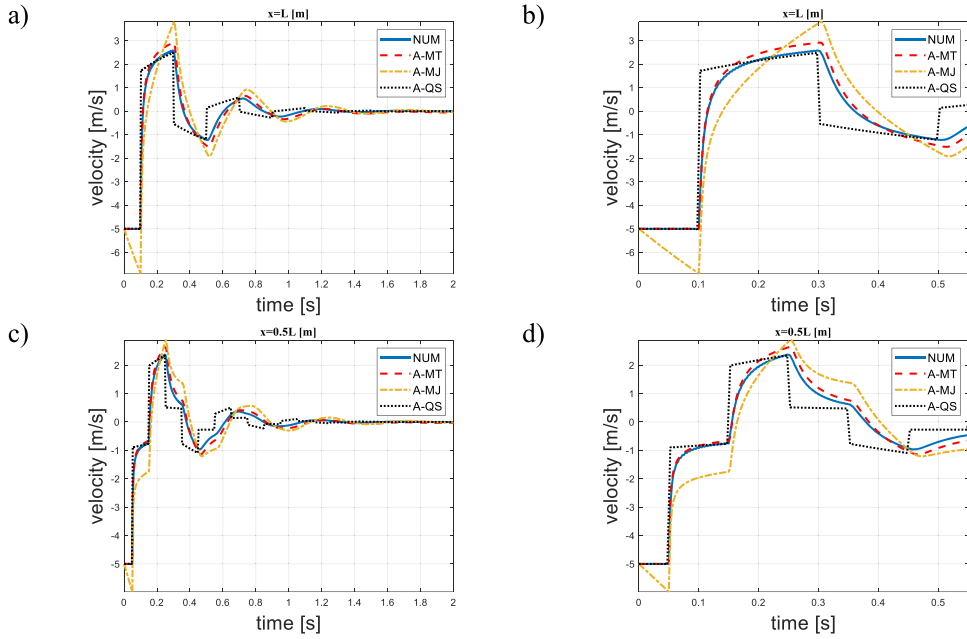


Fig. 9. Velocity time histories for a typical oil flow ($Wh = 0.1$).

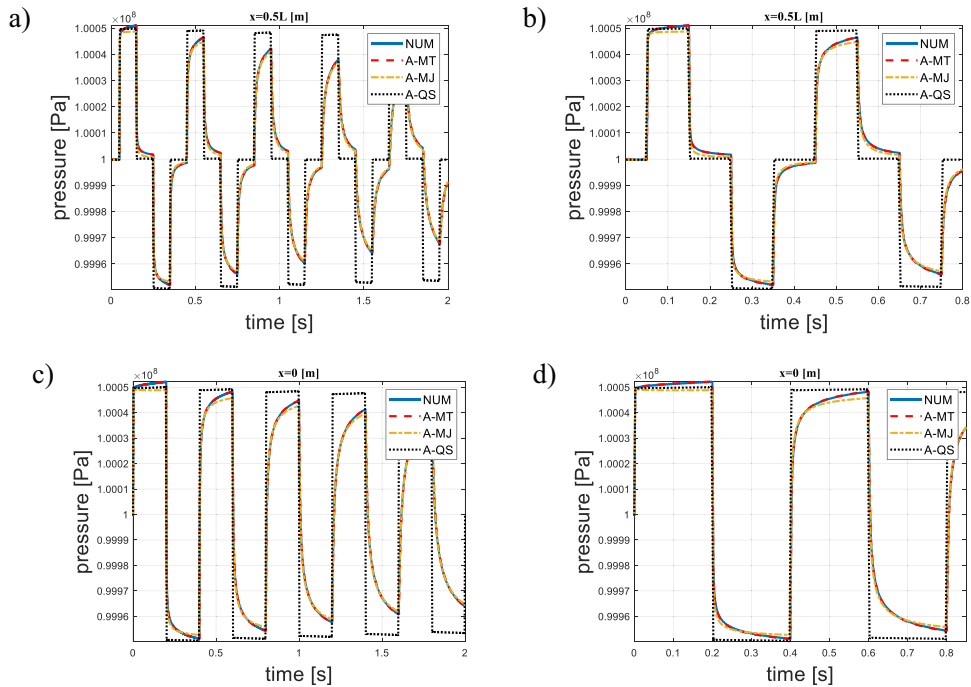


Fig. 10. Pressure time histories for a typical water flow ($Wh = 0.001$).

models can be now used in an extended range of the water hammer number (Wh) i.e. $0 < Wh \leq 0.1$. A further increase of this key dimensionless number results in deviations of the analytical solutions from the experimental and numerical ones. Hence, further research is necessary to determine the correct solutions in a complete range of Wh , in particular in the range of $0.1 < Wh \leq 0.5$, because this is where rapid changes in the dynamic waveforms analyzed in this paper (pressure, bulk velocity and WSS) take place. This range of Wh values predominantly corresponds to oil flows in hydraulic systems. This work also proves the advantage of models based on frequency-dependent friction over quasi-steady models still commonly used in computer applications available on the market. The Mei and Jing model, to which the simulated results were compared, is characterized by correct asymptotic (the final values of the analyzed

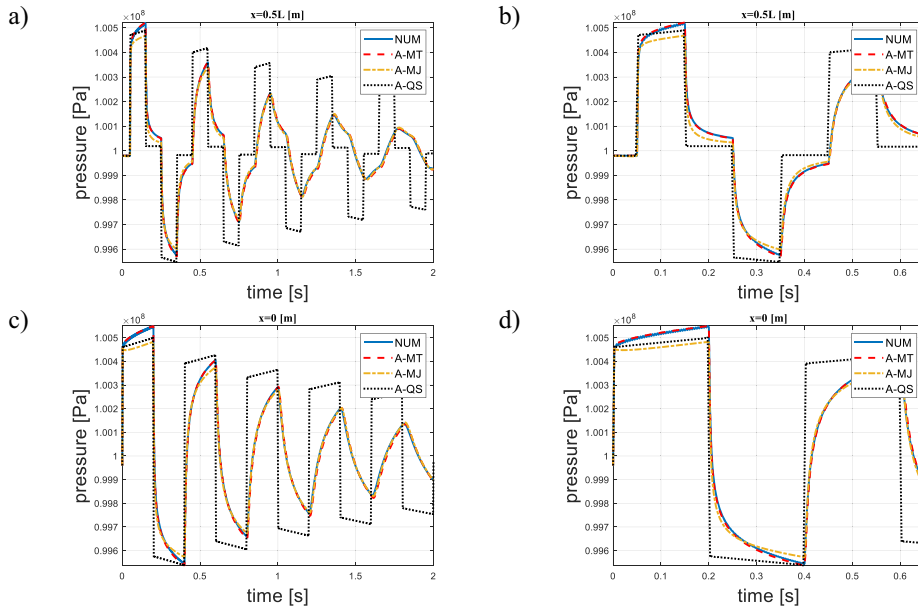


Fig. 11. Pressure time histories for a typical water-oil mixture flow ($Wh = 0.01$).

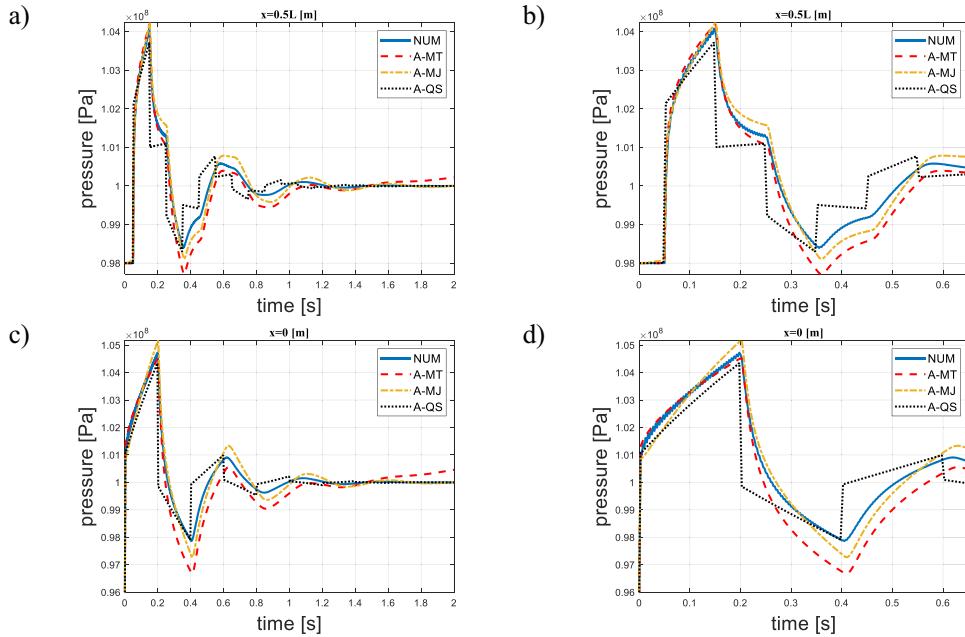


Fig. 12. Pressure time histories for a typical oil flow ($Wh = 0.1$).

parameters: τ , p , v tend to the correct final value), but it has problems with correct prediction of the initial phase of the water hammer event. This defect is particularly visible for bulk velocity and wall shear stress at cross-sections located away from the impact source (at the valve).

Additionally, the conducted research has shown that there is also a possibility of a complete analytical transformation of the dynamic viscosity function, which plays the most important role in the derived transforms of pressure, bulk velocity and WSS - this is a topic of the authors' current research. At the current stage of research, it seems that there is a possibility to develop a complete analytical solution of water hammer equations that will allow for a broad in-depth study of this fascinating phenomenon and allow for further improvements of the existing numerical methods. It should be noted that exist semi-analytical methods for transient laminar pipe flow too in which the inverse Laplace transform can be carried out numerically [61]. New discoveries in turbulence flow theory should be considered in the future research of analytical modeling of transitional and turbulent transient pipe flows [62]. Finally, the

presented considerations are important from the industrial, practical point of view, as the pipeline designers, using the analytical solutions, will be able to protect sensitive sections of pipes so as to prevent their rupture and critical damage of the installation. It will be possible to prevent pipeline failures while still designing the network. With the help of the presented solutions, it will be possible in the near future to monitor sensitive sections of hydraulic pipe networks on a continuous basis.

CRediT authorship contribution statement

Kamil Urbanowicz: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Anton Bergant:** Conceptualization, Software, Formal analysis, Writing – original draft, Writing – review & editing. **Michał Stosiak:** Investigation, Resources, Project administration, Writing – review & editing. **Mykola Karpenko:** Visualization, Data curation, Software. **Marijonas Bogdevičius:** Supervision, Writing – original draft, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A

The original Mei and Jing solution presented in papers [30,31] was not error free. The error was noticed by the main author of this paper during working on conference paper [60]. The problem was related to the “line packing” effect [44] that in original Mei and Jing solutions (for: pressure eq. (5.8); bulk velocity eq. (6.1) and WSS (6.12) from the original paper [30]) was not included. The revised asymptotic solutions, derived by using the multi-scale method (the same technique that was described in detail [30]), have been provided in private correspondence by H. Jing:

$$p^*(x^*, t^*) = \sum_{m=-\infty}^{\infty} \left[e^{(ik_m - c\lambda_m)t^*} \cdot \frac{\sin(k_m(1-x^*))}{ik_m \sin(k_m)} \left(1 + \frac{8\epsilon^2}{ik_m} \right) \right] + 8\epsilon^2 \quad (A1)$$

$$v^*(x^*, t^*) = \sum_{m=-\infty}^{\infty} \left[e^{(ik_m - c\lambda_m)t^*} \cdot \frac{\cos(k_m(1-x^*))}{k_m \sin(k_m)} \left(1 + \frac{8\epsilon^2}{ik_m} \right) \right] \quad (A2)$$

$$\tau^*(x^*, t^*) = - \sum_{m=-\infty}^{\infty} \lambda_m \left[e^{(ik_m - c\lambda_m)t^*} \cdot \frac{\cos(k_m(1-x^*))}{k_m \sin(k_m)} \left(1 + \frac{8\epsilon^2}{ik_m} \right) \right] \quad (A3)$$

where: $\lambda_m = (1 \pm i)\sqrt{\frac{|k_m|}{2}}$ (“+” while $k_m > 0$ and “−” when $k_m < 0$); $\epsilon = \sqrt{Wh}$; $k_m = (m + 0.5)\pi$; $x^* = x/L$; $t^* = (c/L)t$; $v^* = v/v_0$; $p^* = p/(\rho c v_0)$ and $\tau^* = \tau \frac{R\sqrt{Wh}}{\rho v_0}$.

These eqs. (A1–A3) can be further simplified (using the technique proposed in [60]) to the form expressed by eqs. (19–21) free from imaginary unit number.

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