Asymptotic solution of the mathematical model of nonlinear oscillations of absolutely elastic inextensible weightless string

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Abstract. The mathematical model of nonlinear oscillations of weightless string is analyzed. To find an asymptotic solution of the problem, uniformly valid in a long interval of time, an averaged system of integral differential equations has been constructed. A method for constructing special approximations of its solutions is proposed.

Keywords: perturbation methods, averaging, nonlinear waves, resonance, approximations.

1 Introduction

The differential equation $u_{tt} = c_0^2 u_{xx}$ of the transverse oscillations of the absolutely elastic weightless string is presented in nearly all handbooks of mathematical physics. Here, $u(x,t)$ is a string deviation in point x at the time moment t, $c_0 = \sqrt{T/\rho}$ is sound velocity in string material, depending on its tension T and density ρ. However, this linear differential equation turns into a mathematical model of free linear wave oscillations only in the presence of a strongly simplified motion. It is possible to show [1] that with rejecting the small gradient condition, i.e., when $|\,\theta_x| \geq 1$ (here $\theta(x,t)$ is the angle of the string element deviation from the equilibrium position), the equation of string element motion will be as follows:

$$u_{tt} = \frac{c_0^2 u_{xx}}{(1+u_x^2)^{3/2}}. \tag{1}$$

In the case when deviation from the equilibrium is negligible, i.e., $|\,\theta_x| \ll 1$, the equation of string motion (1) turns into the well known linear wave equation.

In paper [2], the mathematical model of nonlinear oscillations of the absolutely elastic weightless string and the equation of motion (1) were analyzed under two important preconditions:

1. Suppose that the sound velocity c is a weakly periodical spatial function

   $$c = c_0(1 + \varepsilon_1 \cos \omega x), \quad \varepsilon_1 \ll 1.$$

   The weakly periodic function is included in sound velocity to consider the possible weak inhomogeneities of the string material or tension.

2. Upon introducing one more small dimensionless parameter ε_2, let us employ the expansion

   $$\frac{1}{\sqrt{(1 + (\varepsilon_2\, u_x)^2)^3}} = 1 - \frac{3}{2}\varepsilon_2^2 u_x^2 + \frac{15}{8}\varepsilon_2^4 u_x^4 + O\big(\varepsilon_2^6\big), \quad \varepsilon_2 \ll 1.$$

Introduction of the small parameter ε_2 allows rejecting the requirement $|\,\theta_x| \ll 1$ and analyzing non-small deviations of the string from the equilibrium $u_x = O(1)$.

With the above preconditions and upon introducing $\tilde{t} = t/c_0$, equation (1) turns into the following equation ($\tilde{t}$ will be again marked as t):

$$u_{tt} - (1 + \varepsilon_1 \cos \omega x)^2 u_{xx}\left(1 - \frac{3}{2}\varepsilon_2^2 u_x^2 + \frac{15}{8}\varepsilon_2^4 u_x^4\right) = 0. \tag{2}$$

2 The model of nonlinear string oscillations

It is important to emphasize, and we shall soon see it, that the classical wave equation is obtained only in case the motion is highly simplified.

Let a strained string with fixed ends be able of freely oscillating in the x, y plane. The linear density of the string material is $\rho = const$, and the tension force module T is a constant value.[1]

We shall consider the string to be massless. This means that its equilibrium position coincides with the x axis, i.e., the effect of gravitation is neglected. More exactly, the gravity force is mach more less than tension force. In the opposite case of heavy string, gravitation should be considered. The equilibrium position of such string is a "chain line" $\sim \cosh y$.

Depending on the context, the weightlessness of the string may be expressed differently. For instance, when the spreading of elastic waves in gases is analyzed and when, in direct approximation, the same wave equation is obtained, the negligible effect of gravitation equals ignoring the barometric pressure. This precondition is valid for rather "thin" layers of gas. The massless flow in hydrodynamics corresponds to the situation with a rather small Galilei number (Ga).

[1] Traditionally, tension force is marked as T ("tension"); we shall differentiate it from the oscillation time which is marked also as T.

Every point x of the string, under the effect of an external stimulus, deviates from the equilibrium: $x \to u(x,t)$, i.e., $u(x,t)$ is an instantaneous deviation of the string in point x at moment t. Let us single out a small fragment $\mathrm{d}x$ of the string with the mass $\rho\,\mathrm{d}x$ (Fig. 1). The string tension force at the ends of the fragment is tangential, therefore the string fragment $\mathrm{d}x$ is pushed from the equilibrium by the transverse resultant of the forces:

$$\mathrm{d}F_y = T\sin\left(\theta + \mathrm{d}\theta\right) - T\sin\theta \simeq T\cos\theta\,\mathrm{d}\theta. \tag{3}$$

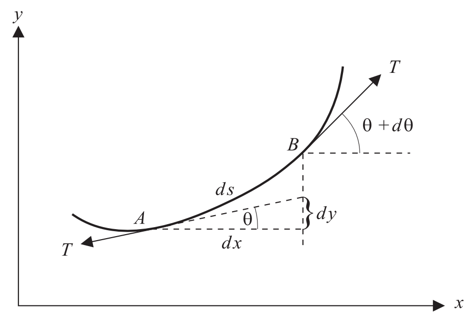

Fig. 1. Fragment of moving string and forces acting on it.

Applying the second law of Newton to a string fragment with the mass $\rho\,\mathrm{d}x$, in the y direction we shall obtain

$$\rho\,\mathrm{d}x\ddot{u} = T\cos\theta\,\mathrm{d}\theta, \tag{4}$$

or

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\cos\theta\frac{\partial\theta}{\partial x}\,. \tag{5}$$

At this point, it should be noted that the moving string deviating from the equilibrium is considered to be absolutely elastic. This means that the tension force T satisfies Hook's law of elasticity. In the opposite case of non-elastic deformations, plastic deformations of the string should be accounted for, and its oscillations will be much more complicated.

Another property of the string, related to its deformation, is its non tensility: we shall consider its length to be stable even when the string deviates from the equilibrium and its deformation satisfies Hook's law. This condition, which at first sight seems to contradict the existence of elastic deformations, means that the string length l is an invariant of motion. In this case, there is no contradiction, because the elasticity force depends on the first and the length of the functional on the second degree of the string gradient. Therefore, even under elastic deformations, the length of the oscillating string may be considered to be constant if $\delta l \ll l$.

As follows from Fig. 1, let us benefit from

$$\tan\theta = \frac{\mathrm{d}y}{\mathrm{d}x} \equiv \frac{\partial u}{\partial x} \tag{6}$$

i.e.,

$$\cos\theta = \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right]^{-1/2} \tag{7}$$

and, differentiating by (6), we shall obtain that

$$\frac{\partial\theta}{\partial x} = (\cos\theta)^2 \frac{\partial^2 u}{\partial x^2}. \tag{8}$$

Upon substituting expressions (7) and (8) into equation (5), we obtain the equation of motion (1) of a string fragment, where $c_0 = \sqrt{T/\rho}$ is sound velocity in the string material.

Let us note some of the main properties of the string equation of motion:

- first of all, we see that equation (1) is nonlinear, i.e., for its solution the principle of superposition is invalid; from this, it follows that in the case of equation (1) the known D'Alembert principle cannot be applied;
- the harmonic waves $u(x,t) = A\cos(x \pm c_0 t + \varphi_0)$ are not the solutions of nonlinear equation (1);
- it is rather easy to show that equation (1) has no nontrivial travelling wave solutions;
- in case the deviations from the equilibrium are negligible, $|u_x| \ll 1$ the equation of string motion turns into the well-known wave equation: $u_{tt} = c_0^2 u_{xx}$.

The need for applied solutions of equation (1), on the one hand, and the lack of information on the analytical solutions of this equation, on the other hand, urges the applications of numerical methods.

3 State of problem

In paper [2], the averaged system of equation (2) was constructed without presenting its solution. In the current work, we will analyze the non-resonance case of the problem, whose analysis was studied in paper [3], and the resonance case which has never been analyzed.

We shall analyze equation (2), assuming that the small parameters ε_1 and ε_2 have the following correlation:

$$\frac{3}{16\beta} \cdot \varepsilon_2^2 = \frac{1}{\alpha} \cdot \varepsilon_1 = \varepsilon. \tag{9}$$

Note that our method works also when precondition (9) is not valid, i.e., when $\varepsilon_2^2 \ll \varepsilon_1$ or $\varepsilon_2^2 \gg \varepsilon_1$. This only alleviates the asymptotic analysis of problem (2) because in this case in equation (10) the number of terms of the $O(\varepsilon)$ order diminishes.

Then, from (2) and (9), we get the equation

$$u_{tt} - u_{xx} = \varepsilon\bigl(\alpha\cos(\omega x)u_{xx} - \beta u_x^2\bigr) + O\bigl(\varepsilon^2\bigr). \tag{10}$$

Upon noting

$$r^+ = u_t + u_x \quad \text{and} \quad r^- = u_t - u_x,$$

we rewrite equation (10) into the following equivalent system through the Riemann invariants:

$$\frac{\partial r^\pm}{\partial t} \mp \frac{\partial r^\pm}{\partial x} = \pm\varepsilon\left(\frac{\partial r^+}{\partial x} + \frac{\partial r^-}{\partial x}\right)\cdot\bigl(\alpha\cos\omega x - \beta\bigl(r^+ - r^-\bigr)^2\bigr). \tag{11}$$

The nonperturbated system (11), i.e., when $\varepsilon = 0$, describes two independent waves $r_0^-(x+t)$ and $r_0^+(x-t)$ moving in two different directions. Here, $r^\pm(x)$ are smoothly differentiable functions describing the initial conditions of the problem (11). While trying to construct the direct asymptotic approximation

$$r^\pm(x,t;\varepsilon) = r_0(x \pm t) + \varepsilon r_1^\pm(x,t) + \cdots, \tag{12}$$

we shall encounter the problem of secular terms εt, characteristic of the asymptotic analysis. For this reason, the asymptotics (12) will be valid only when $\varepsilon t \ll 1$, i.e., in a short time interval $t \ll \varepsilon^{-1}$.

The aim of the present work is the construction of asymptotics in the large domain

$$(t,x) \in \left[0, \frac{\tau_0}{\varepsilon}\right] \times (-\infty, +\infty). \tag{13}$$

Here, τ_0 and all other constants are independent of the small parameter ε.

4 Method of averaging

In order to construct the asymptotic solution of the system (11) uniformly valid in the region (13), we must annul the secular terms εt in the explicit expansion. We apply the principles of two scales and averaging along characteristics. For the details of the averaging scheme, see [5] as well as [6, 7]. For the mathematical substantiation of the method, see [4].

Let us note the slow time

$$\tau = \varepsilon t$$

and the fast characteristic variables

$$y^\pm = x \pm t.$$

We shall look for the approximation of the asymptotic problem (11) in the form [4–7]:

$$r\pm(x,t;\varepsilon) \approx R^{\pm}\big(\tau, y^{\pm}\big).$$

The functions $R^{\pm}$ will be searched for by solving the following averaged system:

$$\frac{\partial R^{\pm}}{\partial \tau} = \pm\left\langle \left(\frac{\partial R^{+}}{\partial y^{+}} + \frac{\partial R^{-}}{\partial y^{-}}\right) \cdot \left(\alpha \cos \omega x - \beta\big(R^{+}R^{-}\big)^{2}\right)\right\rangle_{\pm}. \tag{14}$$

The averaging operators $\langle \cdots \rangle_{\pm}$ according to their characteristics are described as

$$\big\langle g\big(\tau, x, y^{+}, y^{-}\big)\big\rangle_{+} \equiv \lim_{S\to+\infty} S^{-1} \int_{0}^{S} g\big(\tau, y^{+} - s, y^{+}, y^{+} - 2s\big)\, \mathrm{d}s, \tag{15}$$

$$\big\langle g\big(\tau, x, y^{+}, y^{-}\big)\big\rangle_{-} \equiv \lim_{S\to+\infty} S^{-1} \int_{0}^{S} g\big(\tau, y^{-} + s, y^{-} + 2s, y^{-}\big)\, \mathrm{d}s. \tag{16}$$

From descriptions (15) and (16) there follow the properties of the averaging operators [4]:

$$\left\langle \frac{\partial R^{\pm}}{\partial y^{\pm}} \right\rangle_{\mp} = 0, \quad \left\langle \frac{\partial R^{\pm}}{\partial y^{\pm}} R^{\pm} \right\rangle_{\mp} = 0, \quad \left\langle \frac{\partial R^{\pm}}{\partial y^{\pm}} \big(R^{\pm}\big)^{2} \right\rangle_{\mp} = 0, \tag{17}$$

$$\left\langle \frac{\partial R^{\mp}}{\partial y^{\mp}} \cos \omega x \right\rangle_{\mp} = 0, \quad \left\langle \frac{\partial R^{\mp}}{\partial y^{\mp}} \big(R^{\pm}\big)^{2} \right\rangle_{\mp} = \frac{\partial R^{\mp}}{\partial y^{\mp}} \left\langle \big(R^{\pm}\big)^{2} \right\rangle_{\mp}, \tag{18}$$

which are valid for the functions $R^{\pm}(\tau, y)$ (τ_0 is a positive constant) described in the region

$$(\tau, y) \in [0, \tau_0] \times (-\infty, +\infty).$$

Let us substitute formulas (17) and (18) into system (14):

$$\frac{\partial R^{\pm}}{\partial \tau} \pm \beta\big(R^{\pm}\big)^{2} \frac{\partial R^{\pm}}{\partial y^{\pm}} = \pm\alpha \left\langle \frac{\partial R^{\mp}}{\partial y^{\mp}} \cos \omega x \right\rangle_{\pm} \pm \beta \frac{\partial R^{\pm}}{\partial y^{\mp}} \big\langle \big(R^{\mp}\big)^{2} \big\rangle_{\pm}. \tag{19}$$

5 Averaging system

Suppose that system (11) has been complemented with the initial conditions

$$r^{+}(x, 0) = r_0^{+}(x), \quad r^{-}(x, 0) = r_0^{-}(x), \quad x \in (-\infty, +\infty). \tag{20}$$

The asymptotic solution is obtained by solving the averaged system (19), (20). In separate cases, $e.g.$ when the functions r^{+} and r^{-} disappear in the infinity,

$$\lim_{|x|\to\infty} r^{\pm}(x) = 0, \tag{21}$$

system (19) is split into two independent equations of simple nonlinear waves:

$$\frac{\partial R^{\pm}}{\partial \tau} \pm \beta\big(R^{\pm}\big)^2 \frac{\partial R^{\pm}}{\partial y^{\pm}} = 0. \tag{22}$$

Note that the obtained equation (22) may be transformed into the classical Hopf equation (also called the Riemann or Euler equation) $w_t + ww_x = 0$ by employing the substitution $w(x,t) = \beta(R^{\pm}(x,t))^2$.

More complicated is the periodical problem. If functions (20) are periodical with the period 2π and ω is an integer, the averaging operators (15) and (16) may be substituted by integrals in a finite interval, and the averaged system (19) is presented as

$$\begin{aligned}
\frac{\partial R^{+}}{\partial \tau} + \beta\cdot\big(R^{+}\big)^2\frac{\partial R^{+}}{\partial y^{+}} &= \frac{\alpha}{2\pi}\int_0^{2\pi}\frac{\partial R^{-}(\tau, y^{+}-2s)}{\partial y^{-}}\cos\big(\omega\big(y^{+}-s\big)\big)\,\mathrm{d}s \\
&\quad - \frac{\partial R^{+}}{\partial y^{+}}\cdot\frac{\beta}{2\pi}\int_0^{2\pi}\big(R^{-}\big)^2\big(\tau, y^{+}-2s\big)\,\mathrm{d}s, \\
\frac{\partial R^{-}}{\partial \tau} - \beta\cdot\big(R^{-}\big)^2\frac{\partial R^{-}}{\partial y^{-}} &= -\frac{\alpha}{2\pi}\int_0^{2\pi}\frac{\partial R^{+}(\tau, y^{-}+2s)}{\partial y^{+}}\cos\big(\omega\big(y^{-}+s\big)\big)\,\mathrm{d}s \\
&\quad + \frac{\partial R^{-}}{\partial y^{-}}\cdot\frac{\beta}{2\pi}\int_0^{2\pi}\big(R^{+}\big)^2\big(\tau, y^{-}+2s\big)\,\mathrm{d}s.
\end{aligned} \tag{23}$$

Let us supplement system (23) with periodical initial conditions:

$$R^{\pm}\big(\tau, y^{\pm}\big)\big|_{\tau=0} = R_0^{\pm}\big(y^{\pm}\big) \equiv R_0^{\pm}\big(y^{\pm}+2\pi\big). \tag{24}$$

Here α, β, ω are constant parameters, $\tau = \varepsilon t$ is the slow time, and $y^{\pm} = x \mp t$ are the rapid characteristic variables. When the functions $R_0^{\pm}$ are smoothly differentiated, there exists a positive constant τ_0 which makes the problem (23), (24) to have only one smoothly differentiated (as many times as $R_0^{\pm}$) solution $R^{\pm}(\tau, y^{\pm})$ periodical according to $y^{\pm}$ in the domain $[0,\tau_0]\times(-\infty,+\infty)$ [4].

Let us note that systems like (23), (24) appear while applying the method of averaging in models of the resonance interaction of nonlinear waves. In many cases such systems are left unsolved as a separate problem for finding asymptotics [2,8–11]. In [5–7], problems similar to (23), (24) were solved by numerical methods.

6 Approximation of solutions

The aim of the present part is to construct for the solution of (23) and (24) an approximation of the following form:

$$R_N^{\pm}\big(\tau, y^{\pm}\big) = a_0^{\pm}(\tau) + \sum_{k=1}^{N} a_k^{\pm}(\tau)\cos\big(ky^{\pm}\big) + b_k^{\pm}(\tau)\sin\big(ky^{\pm}\big). \tag{25}$$

Let us substitute (25) into system (23). We obtain a new system of differential equations. We show a fragment of the Maple program, when terms are grouped to the similar harmonics $\cos(ky^{\pm})$ and $\sin(ky^{\pm})$:

for k to n do $a_k^{\pm}(\tau) := \mathrm{coeff}(R^{\pm}(\tau, y^{\pm}), \cos(k \cdot y^{\pm}))$ end do;
for k to n do $b_k^{\pm}(\tau) := \mathrm{coeff}(R^{\pm}(\tau, y^{\pm}), \sin(k \cdot y^{\pm}))$ end do.

We will look for functions $a_k^{\pm}(\tau), b_k^{\pm}(\tau)$, $\tau \in [0, \tau_0]$ in the form of M-degree polynomials with undefined coefficients:

$$a_k^{\pm}(\tau) = \sum_{i=0}^{M} a_{ki}^{\pm} \tau^i, \quad b_k^{\pm}(\tau) = \sum_{i=1}^{M} b_{ki}^{\pm} \tau^i. \tag{26}$$

Upon substituting phenomena (25), (26) into the obtained system of equations, we shall group the terms near similar τ degrees and note the relations to find the polynom coefficients $a_k^{\pm}, b_k^{\pm}$. The fragment of term grouping in the Maple program looks as follows:

for i to M do
for k to N do $a_{0i}^{\pm} := \mathrm{subs}(\tau = 0, a_k^{\pm})$ end do;
for k to N do $b_{0i}^{\pm} := \mathrm{subs}(\tau = 0, b_k^{\pm})$ end do;
for k to N do $b_{ki}^{\pm} := \mathrm{coeff}(\mathrm{collect}((1/k) \cdot b_k^{\pm}, [\tau^i]), \tau^i)$ end do;
for k to N do $a_{ki}^{\pm} := \mathrm{coeff}(\mathrm{collect}((1/k) \cdot a_k^{\pm}, [\tau^i]), \tau^i)$ end do;
end do.

7 Nonresonance case

Let us analyze a non-resonance case, i.e., when in system (23) value ω is not an integer. Note that the same result will be obtained for $\alpha = 0$:

$$\frac{\partial R^{\pm}}{\partial \tau} \pm \beta \cdot \left(R^{\pm}\right)^2 \frac{\partial R^{\pm}}{\partial y^{+}} = \mp \frac{\partial R^{\pm}}{\partial y^{\pm}} \cdot \frac{\beta}{2\pi} \int_0^{2\pi} \left(R^{\mp}\right)^2 \left(\tau, y^{\pm} \mp s\right) \mathrm{d}s. \tag{27}$$

We shall solve equation (27) when the periodical initial conditions

$$R^{\pm}\left(\tau, y^{\pm}\right)\big|_{\tau=0} = \cos\left(y^{\pm}\right). \tag{28}$$

Note that for function $R^{\pm}(\tau, y)$ (25), valid is the equality

$$\frac{1}{2\pi} \int_0^{2\pi} \left(R^{\mp}\right)^2 (\tau, y)\, \mathrm{d}y = a_0^2(\tau) + \sum_{k=1}^{\infty} a_k^2(\tau) + b_k^2(\tau),$$

i.e., the integrals of system (27) do not depend on $y^{\pm}$. Let us multiply each of equations (27) by R^{+} and R^{-} respectively and integrate by y from 0 to 2π. We obtain:

$$\frac{\partial}{\partial y^{\pm}}\left(\frac{1}{2\pi}\int\limits_{0}^{2\pi}\left(R^{\mp}\right)^{2}\left(\tau, y^{\pm}\mp s\right)\mathrm{d}s\right) = 0. \tag{29}$$

Thus, $\frac{1}{2\pi}\int_0^{2\pi}(R^{\mp})^2(\tau, y^{\pm}\mp s)\,dis = \frac{1}{2\pi}\int_0^{2\pi}(R^{\mp})^2(0, y^{\pm}\mp s)\,dis - const$ and the two equations (27) are independent.

In this case, each solution of the problem (27), (28) may be presented in the form of an inexplicit function:

$$R^{\pm}\left(\tau, y^{\pm}\right) = \cos\left(y^{\pm}\mp\beta\left(\left(R^{\pm}\right)^{2}\pm\left(R^{\mp}\right)^{2}\right)\tau\right). \tag{30}$$

Let us note that (30) is the solution of the problem. The derivative of the function $R^{\pm}(\tau, y^{\pm})$ with respect to τ is

$$\frac{\partial}{\partial\tau}R^{\pm}\left(\tau, y^{\pm}\right) = \pm\frac{\sin(y^{\pm}\mp\beta((R^{\pm})^2+(R^{\mp})^2)\tau)(\beta((R^{\pm})^2\pm(R^{\mp})^2))}{1\mp\sin(y^{\pm}\mp\beta((R^{\pm})^2+(R^{\mp})^2)t)2\beta\tau R^{\pm}}. \tag{31}$$

The derivative by y of the functions $R^{\pm}(\tau, y^{\pm})$ is:

$$\frac{\partial}{\partial y}R^{\pm}\left(\tau, y^{\pm}\right) = -\frac{\sin(y^{\pm}\mp\beta((R^{\pm})^2+(R^{\mp})^2)\tau)}{1\mp\sin(y^{\pm}\mp\beta((R^{\pm})^2+(R^{\mp})^2)\tau)2\beta\tau R^{\pm}}. \tag{32}$$

Substitution (31) and (32) into equations (27) gives identities.

Let us extend the functions $R^{\pm}$ by the Taylor formula. Upon denoting the partial derivatives D_jR of the j-order of the function R by τ, we obtain

$$R^{\pm}\left(\tau, y^{\pm}\right) = \sum_{j=0}^{M}\frac{D_jR^{\pm}(0, y^{\pm})}{j!}\,\tau^{j} + O\left(\tau^{M+1}\right).$$

Upon calculating $D_jR^{\pm}$ order derivatives, we write the formulas of the approximations:

$$\begin{aligned}
R^{\pm}\left(\tau, y^{\pm}\right) = &\left(1-\frac{5}{16}\tau^{2}+\frac{25}{1024}\tau^{4}-\frac{251}{245760}\tau^{6}\right)\cos\left(y^{\pm}\right)\\
&+\left(\mp\frac{3}{4}\tau\pm\frac{37}{384}\tau^{3}\mp\frac{109}{20480}\tau^{5}\pm\frac{321}{1835008}\tau^{7}\right)\sin\left(y^{\pm}\right)\\
&+\left(\frac{21}{32}\tau^{2}-\frac{459}{512}\tau^{4}+\frac{146043}{327680}\tau^{6}\right)\cos\left(3y^{\pm}\right)\\
&+\left(\pm\frac{1}{4}\tau\mp\frac{117}{128}\tau^{3}\pm\frac{2835}{4096}\tau^{5}\mp\frac{112671}{458752}\tau^{7}\right)\sin\left(3\,y^{\pm}\right)\\
&+\left(-\frac{5}{32}\tau^{2}+\frac{875}{512}\tau^{4}-\frac{1068281}{294912}\tau^{6}\right)\cos\left(5\,y^{\pm}\right)
\end{aligned}$$

$$\pm\left(\frac{275}{384}\tau^3 \mp \frac{23125}{8192}\tau^5 \pm \frac{3526125}{917504}\tau^7\right)\sin\left(5y^{\pm}\right)$$
$$+\left(\mp\frac{49}{384}\tau^3 \pm \frac{171521}{61440}\tau^5 \mp \frac{2662093}{245760}\tau^7\right)\sin\left(7y^{\pm}\right)$$
$$+\left(-\frac{1715}{2048}\tau^4 + \frac{18583901}{2949120}\tau^6\right)\cos\left(7y^{\pm}\right) + \cdots .$$

In [3], a numerical investigation was reported. The number N of harmonics in formula (25) and the polynom degree M in (26) varied. The obtained results were compared with extensions of formulas (31) and (32). The effect of the number of harmonics and the polynom degree on the function $R^{\pm}$ extension coefficients was examined (N and M were analyzed up to (11)).

In the table are presented results of numerical calculations, in which the function R^+ was found by *Maple*, as well as differences of approximation: $R^{M,N} - R^+$.

$\tau = 0.1$	$y = 0.1$	$y = 0.5$	$y = 1.0$	$y = 2.0$	$y = 3.14$	$y = 4.0$	$y = 4.5$	$y = 6.28$
R^+_{Maple}	0.9988	0.9350	0.6117	−0.3582	−0.9889	−0.7280	−0.2662	0.9886
$R^+_{3,3} - R^+$	0.0015	−0.0006	−0.0002	−0.0012	0.0012	0.0012	−0.0016	0.0012
$R^+_{5,5} - R^+$	0.0000	0.0000	0.0001	0.0001	0.0001	0.0001	−0.0001	0.0001
$R^+_{7,7} - R^+$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

$\tau = 0.2$	$y = 0.1$	$y = 0.5$	$y = 1.0$	$y = 2.0$	$y = 3.14$	$y = 4.0$	$y = 4.5$	$y = 6.28$
R^+_{Maple}	0.9813	0.9784	0.6941	−0.3056	−0.9593	−0.8096	−0.3277	0.9589
$R^+_{3,3} - R^+$	0.0038	−0.0008	−0.0029	−0.0069	0.0015	0.0060	−0.0074	0.0015
$R^+_{5,5} - R^+$	−0.0003	0.0008	−0.0007	0.0001	0.0005	0.0013	−0.0011	−0.0005
$R^+_{7,7} - R^+$	0.0001	0.0002	−0.0002	0.0001	−0.0002	0.0002	−0.0001	0.0002

$\tau = 0.3$	$y = 0.1$	$y = 0.5$	$y = 1.0$	$y = 2.0$	$y = 3.14$	$y = 4.0$	$y = 4.5$	$y = 6.28$
R^+_{Maple}	0.9445	1.0009	0.7882	−0.2566	−0.9153	−0.8911	−0.3986	0.9148
$R^+_{3,3} - R^+$	0.0034	0.0057	−0.0102	−0.0167	0.0031	0.0101	−0.0175	−0.0032
$R^+_{5,5} - R^+$	0.0041	−0.0003	−0.0049	−0.0024	−0.0039	0.0037	−0.0059	0.0039
$R^+_{7,7} - R^+$	0.0038	−0.0011	−0.0026	0.0007	−0.0034	0.0004	−0.0017	0.0034

$\tau = 0.4$	$y = 0.1$	$y = 0.5$	$y = 1.0$	$y = 2.0$	$y = 3.14$	$y = 4.0$	$y = 4.5$	$y = 6.28$
R^+_{Maple}	0.8661	1.0212	0.8883	−0.2105	−0.8442	−0.9487	−0.4847	0.8441
$R^+_{3,3} - R^+$	0.0211	−0.0102	−0.0176	−0.0307	−0.0001	−0.0068	−0.0299	−0.0004
$R^+_{5,5} - R^+$	0.0516	−0.0288	−0.0162	−0.0148	−0.0415	−0.0003	−0.0190	0.0411
$R^+_{7,7} - R^+$	0.0339	−0.0178	−0.0157	−0.0004	−0.0215	−0.0063	−0.0104	0.0211

$\tau = 0.5$	$y = 0.1$	$y = 0.5$	$y = 1.0$	$y = 2.0$	$y = 3.14$	$y = 4.0$	$y = 4.5$	$y = 6.28$
R^+_{Maple}	0.6369	1.1317	0.9731	−0.1678	−0.6875	−0.9249	−0.5954	0.6894
$R^+_{3,3} - R^+$	0.1645	−0.1447	−0.0050	−0.0477	−0.0697	−0.0992	−0.0387	−0.0671
$R^+_{5,5} - R^+$	0.2784	−0.1790	−0.0318	−0.0500	−0.1925	−0.0420	−0.0459	0.1898
$R^+_{7,7} - R^+$	0.1682	−0.1048	−0.0609	−0.0171	−0.0856	−0.0501	−0.0397	0.0834

The finding has been that $N = 7$ and $M = 7$ are sufficient to ensure the graphical precision. We will show how the graph of the function $R^{\pm}$ approximations looks in the case of non-resonance (i.e., when ω is not an integer) of different values at the slow time.

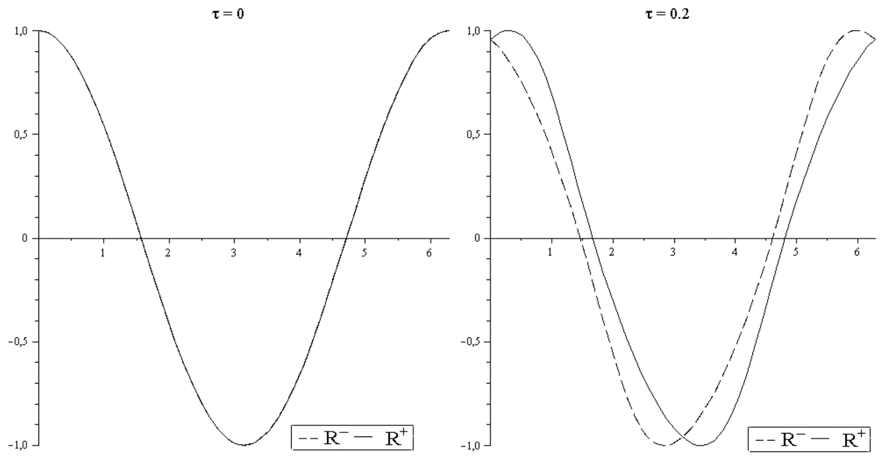

Fig. 2. Approximations of $R^{\pm}$ for $\tau = 0$ and $\tau = 0.2$.

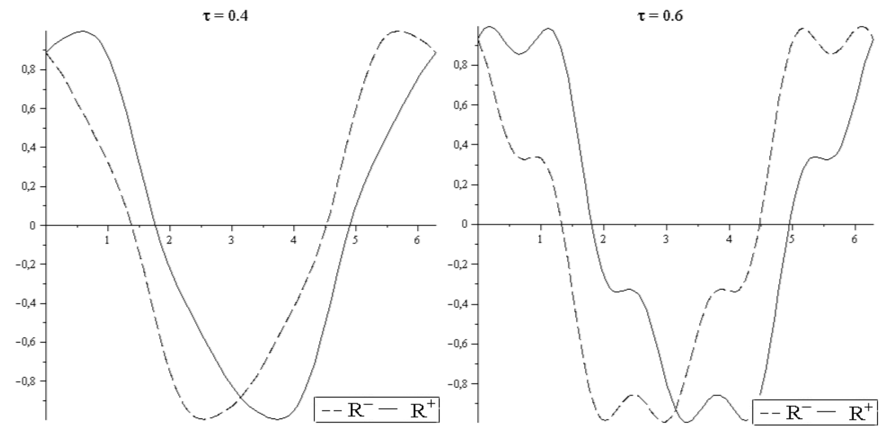

Fig. 3. Approximations of $R^{\pm}$ for $\tau = 0.4$ and $\tau = 0.6$.

We see that with increasing τ_0 the wave amplitude also increases. From equality (30) we see also that $|R^{\pm}| \leqslant 1$. Therefore the obtained formulas are applicable for calculating the functions $R^{\pm}$ only in cases of $\tau \in [0, \tau_0]$ when this condition is satisfied.

Exactly when $\tau \in [0, \tau_0]$, the problem has a classical continuous differential solution (see [4]). The approximation constructed in our work is applicable when $\tau \leq \tau^0 \leq \tau_0$.

The accuracy of this approximation (see the calculation table on p. 316) is inversely proportional to the value of τ^0. In all cases, τ^0 is a constant independent of the small parameter ε. Therefore, the proposed asymptotical solution is uniformly valid in the large area $(t,x) \in [0, \frac{\tau^0}{\varepsilon}] \times R$.

In the case under analysis, $\tau^0 \approx 0.6$.

8 Resonance case

Now, let us proceed to the case of resonance, i.e., when in system (23) ω is an integer. We shall solve this system by constructing approximations (25) and (26) with the aid of the software written in the Maple medium.

Let us investigate the case when the periodical initial conditions are

$$R^+(\tau,y)|_{\tau=0} = \sin\bigl(y^+\bigr), \quad R^-(\tau,y)|_{\tau=0} = 0. \tag{33}$$

We expand functions $R^\pm(\tau,y^\pm)$ to Fourier series and insert into our system. We obtain:

$$\begin{aligned}
&\frac{\partial a_0^\pm(\tau)}{\partial\tau} + \frac{\partial a_1^\pm(\tau)}{\partial\tau}\cos\bigl(y^\pm\bigr) + \frac{\partial b_1^\pm(\tau)}{\partial\tau}\sin\bigl(y^\pm\bigr) \\
&\quad + \frac{\partial a_2^\pm(\tau)}{\partial\tau}\cos\bigl(2y^\pm\bigr) + \frac{\partial b_2^\pm(\tau)}{\partial\tau}\sin\bigl(2y^\pm\bigr) + \cdots \\
&\quad \pm\beta\bigl(a_0^\pm(\tau) + a_1^\pm(\tau)\cos\bigl(y^\pm\bigr) + b_1^\pm(\tau)\sin\bigl(y^\pm\bigr) + a_2^\pm(\tau)\cos\bigl(2y^\pm\bigr) \\
&\qquad + b_2^\pm(\tau)\sin\bigl(2y^\pm\bigr) + \cdots\bigr)^2\bigl(-a_1^\pm(\tau)\sin\bigl(y^\pm\bigr) + b_1^\pm(\tau)\cos\bigl(y^\pm\bigr) \\
&\qquad\qquad -2a_2^\pm(\tau)\sin\bigl(2y^\pm\bigr) + 2b_2^\pm(\tau)\cos\bigl(2y^\pm\bigr) + \cdots\bigr) \\
&\quad \pm\frac{\alpha}{2\pi}\int_0^{2\pi}\bigl(-a_1^\mp(\tau)\cos\bigl(y^\pm \mp 2s\bigr) + b_1^\mp\sin\bigl(y^\pm \mp 2s\bigr) - 2a_2^\mp\cos\bigl(2y^\pm \mp 2s\bigr) \\
&\qquad + 2b_2^\mp(\tau)\sin\bigl(2y^\pm \mp 2s\bigr) + \cdots\bigr)\cos\bigl(\omega\bigl(y^\pm \mp s\bigr)\bigr)\,\mathrm{d}s \\
&\quad - \bigl(-a_1^\pm(\tau)\sin\bigl(y^\pm\bigr) + b_1^\pm(\tau)\cos\bigl(y^\pm\bigr) \\
&\qquad - 2a_2^\pm(\tau)\sin\bigl(2y^\pm\bigr) + 2b_2^\pm(\tau)\cos\bigl(2y^\pm\bigr) + \cdots\bigr) \\
&\qquad \times\biggl(\pm\frac{\beta}{2\pi}\int_0^{2\pi}\bigl(a_0^\mp(\tau) + a_1^\mp(\tau)\cos\bigl(y^\pm \mp 2s\bigr) + b_1^\mp(\tau)\sin\bigl(y^\pm \mp 2s\bigr) \\
&\qquad\qquad + a_2^\mp(\tau)\cos\bigl(2y^\pm \mp 2s\bigr) + b_2^\mp(\tau)\sin\bigl(2y^\pm \mp 2s\bigr) + \cdots\bigr)^2\mathrm{d}s\biggr).
\end{aligned}$$

We obtain a new scheme in which we put together terms with the same indexes and thus obtain a system of ordinary differential equations. The formulas are rather cumbersome, so we don't show them, and the Maple program is used to find the coefficients.

We show the results of calculation when in formula (25) $M = 3$, $M = 5$ and $M = 7$:

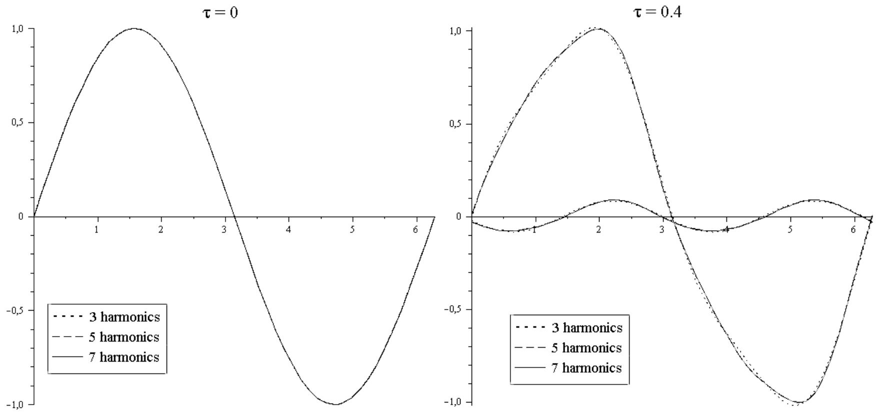

Fig. 4. Graphs of $R^{\pm}_{7,3}$, $R^{\pm}_{7,5}$, $R^{\pm}_{7,7}$ for $\tau = 0$ and $\tau = 0.4$.

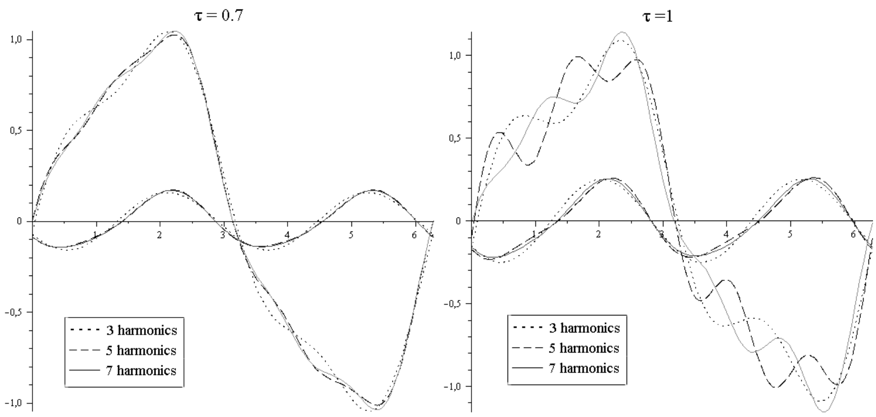

Fig. 5. Graphs of $R^{\pm}_{7,3}$, $R^{\pm}_{7,5}$, $R^{\pm}_{7,7}$ for $\tau = 0.7$ and $\tau = 1$.

From the above graphs we see also that to insure the graphical accuracy of calculations $N = 7$ is sufficient. It is possible to show that the same holds also for the polynom degree $M = 7$.

Note that because of the resonance interaction of the waves, in the course of time the wave R^- appears, although at the time moment $\tau = 0$, R^- was a zero function. It is notable also that in the non-resonance case the wave R^- does not appear, i.e., at all τ values we have $R^- \equiv 0$.

9 Profile of string

Let us show, by the method of asymptotic integration presented in this paper, the profile variation dynamics of an absolutely elastic weightless string. Because we have formulas for the Riemann invariants

$$r^+ = u_t + u_x \quad \text{and} \quad r^- = u_t - u_x,$$

we obtain that

$$u_x = \frac{r^+ - r^-}{2}.$$

We return to the fast characteristic variables:

$$y^- = x - t, \quad y^+ = x + t$$

and write the string profile equation

$$u = \int \frac{r^+ - r^-}{2}\, \mathrm{d}x + C.$$

We shall analyze string profile variation dynamics for non-resonance and resonance cases when $N = 7$, $M = 7$ and $\beta = 0.35$, $\omega = 1$, $\alpha = 3.0$. At the initial moment of time, we have the Riemann invariants $r^+(0, x) = \sin x$, $r^-(0, x) = 0$, and the corresponding string profile is $u(0, x) = \frac{1}{2}\sin x$. We show approximation graphs of the functions $u(t, x; \varepsilon)$, $r^+(t, x; \varepsilon)$, $r^-(t, x; \varepsilon)$ when $\varepsilon = 0.01$ and t varies from 10 to 100 (t values are indicated).

The left side shows the functions in the non-resonance case. Note that the wave r^- in this case does not appear, i.e., $r^- \equiv 0$. On the right side, the same functions are shown for the case of resonance.

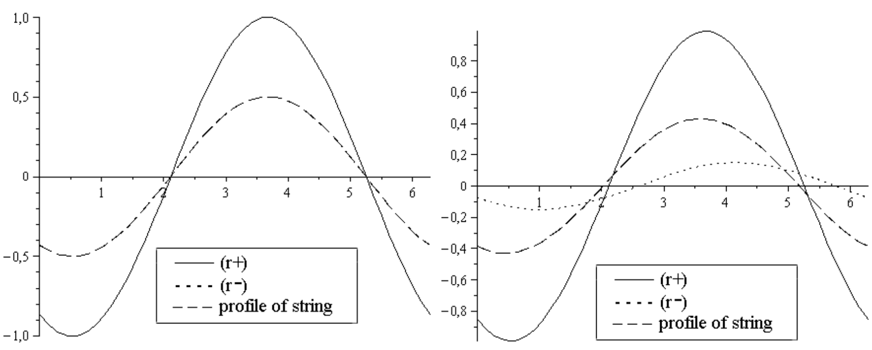

Fig. 6. Profiles of string in non-resonance and resonance cases for $t = 10$.

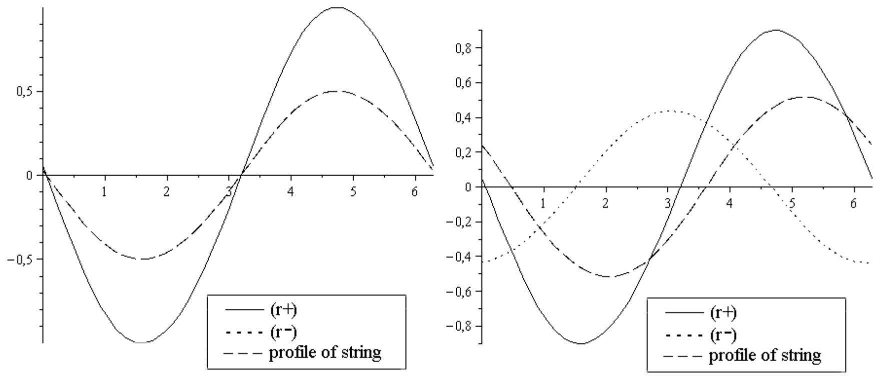

Fig. 7. Profiles of string in non-resonance and resonance cases for $t = 30$.

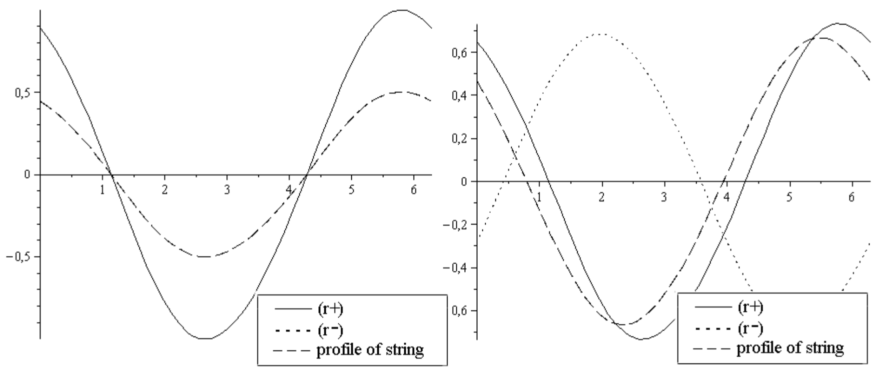

Fig. 8. Profiles of string in non-resonance and resonance cases for $t = 50$.

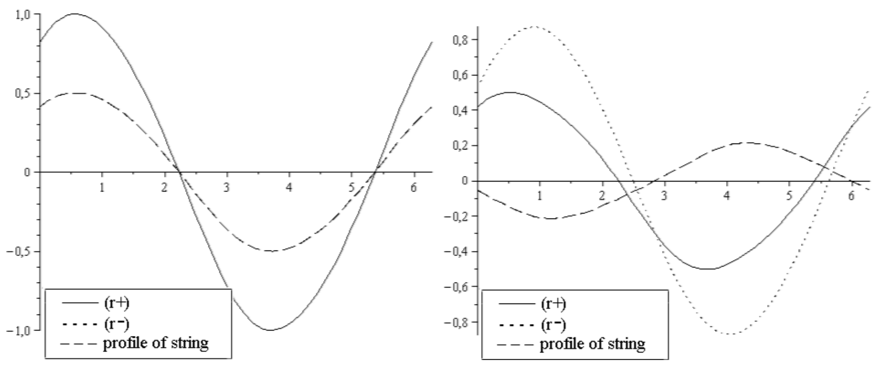

Fig. 9. Profiles of string in non-resonance and resonance cases for $t = 70$.

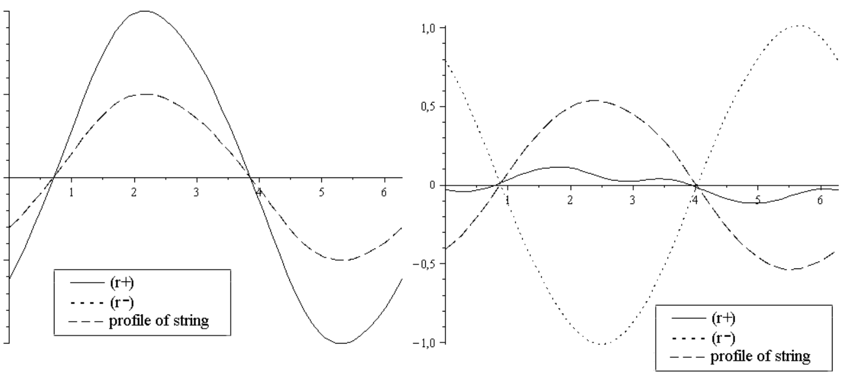

Fig. 10. Profiles of string in non-resonance and resonance cases for $t = 100$.

10 Conclusions

Thus, the obtained results are as follows.

An integral differential system of averaged equations has been constructed for modelling nonlinear oscillations of the absolutely elastic weightless string. To solve this system, the Maple software has been compiled, which allows constructing the solution approximations of a special form.

In the non-resonance case, an exact solution is possible, and this allowed testing the program. The accuracy of the approximation under construction depends on the number of harmonics and the degree of the polynoms that approximate the extension. The effect of these parameters is analyzed by numerical experiments.

The calculations presented in the paper deal with the non-resonance case. In future, we intend to carry out a theoretical investigation and to establish whether the approximation error of calculations presented in the paper depends on the parameters τ_0, M, N.

For the resonance case, calculations of similar problems have been presented in [5–7], where finite difference schemes were proposed. It would be interesting to compare the results obtained by this method with numerical simulations.

Calculations have been performed to show string profile variations in a long time interval for the resonance and non-resonance cases.

Interestingly, the obtained asymptotic formulas, e.g. (23), allow recalculating the values of functions for other values of ε and t. For example, the same graphs will be obtained when $\varepsilon = 0.001$ and the corresponding values of t will be $t = 100$, $t = 300, \ldots, t = 1000$ or $\varepsilon = 0.0001$ and $t = 1000$, $t = 3000, \ldots, t = 10000$.

Acknowledgment

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