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Asymptotic Expansions for the Distribution and Density Functions of the Quadratic Form of a Stationary Gaussian Process in the Large Deviation Cramer Zone

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Abstract

The work considers the asymptotic expansions in the large deviation Cramer zone for the distribution and its density functions of the quadratic form of a stationary Gaussian sequence. To this end the general authors lemma [1], [3] for an arbitrary random variable with regular behaviour of its cumulants is used.

Keywords: distribution, density and characteristic function, cumulant, asymptotic expansion, large deviation.

1 Formulation of the results

Let $\{X_t, t = 1, 2, \dots\}$ be a real stationary Gaussian sequence with mean $\mathbf{E}X_t = 0$ and the covariance matrix (c.m.)

$$R = \left[\mathbf{E}X_s X_t\right]_{s=\overline{1,n}}^{t=\overline{1,n}}, \qquad \det R \neq 0. \tag{1.1}$$

Denote

$$\zeta_n = \sum_{s,t=1}^{n} a_{s,t} X_s X_t, \qquad A = \left[a_{s,t}\right]_{s=\overline{1,n}}^{t=\overline{1,n}} \tag{1.2}$$

where, without loss of generality, we can suppose the matrix to be symmetric. A non-symmetric matrix $\widetilde{A} = \left[\widetilde{a}_{s,t}\right]_{s=\overline{1,n}}^{t=\overline{1,n}}$ can be reduced to a symmetric matrix A, where $a_{s,t} = \frac{1}{2}(\widetilde{a}_{s,t} + \widetilde{a}_{t,s})$ and $a_{s,t} = a_{t,s}$.

We denote by $\mu_1, \mu_2, \ldots, \mu_n$, a spectrum of eigenvalues of matrix RA obtained in the solution of the n^{th} degree algebraic equation $\det\left(A - \mu R^{-1}\right) = 0$.

We know that the distribution of a r.v. ζ_n defined by equality (1.2) is the same as that of the r.v.

$$\eta_n = \sum_{j=1}^{n} \mu_j Y_j^2, \tag{1.3}$$

where Y_j , $j = \overline{1,n}$ are independent Gaussian r.v' s with $\mathbf{E}Y_j = 0$ and $\mathbf{D}Y_j = \mathbf{E}Y_j^2 = 1$. Then

$$\mathbf{E}\zeta_n = \mathbf{E}\eta_n = \sum_{j=1}^{n} \mu_j\,, \quad B_n^2 = \mathbf{D}\zeta_n = \mathbf{D}\eta_n = 2\sum_{j=1}^{n} \mu_j^2 \tag{1.4}$$

Denote by

$$\widetilde{\zeta}_n = B_n^{-1}(\zeta_n - \mathbf{E}\zeta_n)\,, \quad F_{\widetilde{\zeta}_n}(x) = \mathbf{P}\big(\widetilde{\zeta}_n < x\big)\,, \quad p_{\widetilde{\zeta}_n}(x) = \frac{d}{dx} F_{\widetilde{\zeta}_n}(x) \tag{1.5}$$

the distribution and the density function of the r.v. $\widetilde{\zeta}_n$ and by

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,dy\,, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}x^2\right\} \tag{1.6}$$

the (0,1) – normal distribution and its density, respectively.

In order to obtain asymptotic expansions of the distribution function $F_{\widetilde{\zeta}_n}(x)$ and its density $p_{\widetilde{\zeta}_n}(x)$ of the r.v. $\widetilde{\zeta}_n$, defined by equality (1.5), in large deviation zones, according to the general lemmas obtained by the author in [1], [3], one must have the estimates of the k^{th} order cumulants of the r.v. η_n

$$\Gamma_k\big(\eta_n\big) := \frac{1}{i^k}\frac{d^k}{dt^k} \ln f_{\eta_n}(t)\Big|_{t=0}, \qquad k = 1, 2, \ldots, \tag{1.7}$$

where $f_\xi(t) = \mathbf{E}\exp\{it\xi\}$ is the characteristic function of the r.v. ξ.

Let $Z_j := \mu_j Y_j^2$, $j = 1, 2, \ldots, n$. Recalling that Y_j – (0,1) are normal independent r.v., we get

$$f_{Z_j}(t) = \mathbf{E}e^{itZ_j} = f_{Y_j^2}\big(\mu_j t\big) = \big(1-2i\mu_j t\big)^{-1/2}, \tag{1.8}$$

$$f_{\eta_n}(t) = \prod_{j=1}^{n} \big(1-2i\mu_j t\big)^{-1/2}. \tag{1.9}$$

Then, by the definition of $\Gamma_k(\eta_n)$ and by equality (1.7), we obtain

$$\Gamma_k(\eta_n) = 2^{k-1}(k-1)!\sum_{j=1}^{n} \mu_j^k, \qquad k = 1, 2, \ldots. \tag{1.10}$$

Taking into account that

$$\Gamma_1\big(\eta_n - \mathbf{E}\eta_n\big) = 0, \qquad \Gamma_k\big(\eta_n - \mathbf{E}\eta_n\big) = \Gamma_k\big(\eta_n\big), \qquad k = 2, 3, \ldots$$

we get

$$\begin{aligned} \Gamma_k\big(\widetilde{\zeta}_n\big) &= \Gamma_k\Big(\big(\zeta_n - \mathbf{E}\zeta_n\big)/B_n\Big) = \Gamma_k\big(\eta_n\big)/B_n^k = \\ &= 2^{k-1}(k-1)!\sum_{j=1}^{n} \mu_j^k \Big/ \Big(2\sum_{j=1}^{n} \mu_j^2\Big)^{k/2}, k = 2, 3, \ldots \end{aligned} \tag{1.11}$$

Hence we obtain the following estimate of the k^{th} order cumulant $\Gamma_k\big(\widetilde{\zeta}_n\big)$ of the r.v. $\widetilde{\zeta}_n$:

$$\Big|\Gamma_k\big(\widetilde{\zeta}_n\big)\Big| \le (k-1)!/\Delta_n^{k-2}, \qquad k = 2, 3, \ldots, \tag{1.12}$$

where

$$\Delta_n = B_n/(2\max_{1\le j\le n}|\mu_j|) = \big(2\sum_{j=1}^{n}\mu_j^2\big)^{1/2}/(2\max_{1\le j\le n}|\mu_j|). \tag{1.13}$$

Let

$$B = RA = \big[b_{s,t}\big]_{s=\overline{1,n}}^{t=\overline{1,n}}, \qquad P = \max_{1\le s\le n}\sum_{t=1}^{n}|b_{s,t}|, \qquad Q = \max_{1\le t\le n}\sum_{s=1}^{n}|b_{s,t}|.$$

where matrices R and A are defined by equalities (1.1) and (1.2), respectively. Note that

$$\max_{1\le j\le n}|\mu_j| \le \max\{P, Q\}. \tag{1.14}$$

Next, let

$$\Delta_n^* = c_o\Delta_n\,, c_o = (1/6)\big(\sqrt{2}/6\big), T_n = (1/12)\big(1 - x/\Delta_n^*\big)\Delta_n^*\,, \tag{1.15}$$

θ_i, $i = 1, 2, \ldots$, stand for quantities not exceeding a unit in absolute value.

Theorem 1 *For the distribution function* $F_{\widetilde{\zeta}_n}(x)$ *of the r.v.* $\widetilde{\zeta}_n$ *defined by equality* (1.5) *in the interval*

$$0 \le x < \Delta_n^*\,,$$

for each integer l, $l \ge 3$, *the equality*

$$\frac{1 - F_{\widetilde{\zeta}_n}(x)}{1 - \Phi(x)} = \exp\big\{L_n(x)\big\}\bigg\{\frac{\psi(x)}{\psi(u_n)}\Big(1 + \sum_{\nu=1}^{l-3} L_{\nu,n}(u_n)\Big) +$$

$$+\theta_1(x+1)\Big(\frac{c_1(l)}{\Delta_n^{l-2}} + \frac{144\sqrt{2}e^2}{\big(1 - x/\Delta_n^*\big)}\frac{2\max\limits_{1\le j\le n}|\mu_j|}{\prod_{k=1}^4 |\mu_{i_k}|^{1/4}}e^{-T_n^2/5}\Big)\bigg\} \tag{1.16}$$

holds.

Here

$$L_n(x) = \sum_{k=3}^{\infty} \lambda_{k,n} x^k. \tag{1.17}$$

The coefficients $\lambda_{k,n}$ (expressed by cumulants of the r.v. $\widetilde{\zeta}_n$) are found by the formula

$$\lambda_{k,n} = -b_{k-1,n}/k\,, \tag{1.18}$$

where $b_{k,n}$ are determined successively from the equations

$$\sum_{r=1}^{j} \frac{1}{r!}\Gamma_{r+1}(\widetilde{\zeta}_n) \sum_{\substack{j_1+\cdots+j_r=j\\ j_i\ge 1}} \prod_{i=1}^{r} b_{j_i,n} = \begin{cases} 1 & j=1\\ 0 & j=2,3,\end{cases} \tag{1.19}$$

In particular,

$$\begin{aligned}
\lambda_{3,n} &= (1/3)\Gamma_3(\widetilde{\zeta}_n),\\
\lambda_{4,n} &= (1/24)\big(\Gamma_4(\widetilde{\zeta}_n) - 3\Gamma_3^2(\widetilde{\zeta}_n)\big),\\
\lambda_{5,n} &= (1/120)\big(\Gamma_5(\widetilde{\zeta}_n) - 10\Gamma_3(\widetilde{\zeta}_n)\Gamma_4(\widetilde{\zeta}_n) + 15\Gamma_3^3(\widetilde{\zeta}_n)\big), \ldots.
\end{aligned}$$

Here the k^{th} order cumulant $\Gamma_k(\widetilde{\zeta}_n)$, $k = 3, 4, \dots$, is expresed by formula (1.11). For the coefficients $\lambda_{k,n}$, the estimate

$$|\lambda_{k,n}| \leq (2/k)(16/\Delta_n)^{k-2}, k = 3, 4, \dots \tag{1.20}$$

holds, and therefore

$$L_n(x) \leq (x^2/2)\big(x/(x+8\Delta_n^*)\big), \qquad L_n(-x) \geq -\big(x^3/(3\Delta_n^*)\big).$$

The function $\psi(x)$ has the following representation

$$\psi(x) = \varphi(x)/\big(1-\Phi(x)\big), \tag{1.21}$$

where $\Phi(x)$ is the N(0,1) normal distribution with density $\varphi(x)$. The quantity

$$u_n = u_n(x) = x\big(1+\sum_{k=1}^{l-3} c_{k,n}x^k+\theta c^*(l)(x/\Delta_n)^{l-2}\big), \tag{1.22}$$

where $c^*(l) = 736\, l(l-1)(7/2)^{l-2}$, and the coefficients $c_{k,n}$ are expressed by the cumulants of the r.v. $\widetilde{\zeta}_n$ and found by formula (11) [3]. In particular,

$$\begin{aligned}
c_{1,n} &= 0, \\
c_{2,n} &= (1/24)\big(2\Gamma_4(\widetilde{\zeta}_n) - 3\Gamma_3^2(\widetilde{\zeta}_n)\big), \\
c_{3,n} &= (1/24)\big(\Gamma_5(\widetilde{\zeta}_n) - 6\Gamma_3(\widetilde{\zeta}_n)\Gamma_4(\widetilde{\zeta}_n) + 6\Gamma_3^3(\widetilde{\zeta}_n)\big), \dots
\end{aligned}$$

Polynomials $L_{\nu,n}(u_n)$ are determined by relation (104) [3]. In particular,

$$\begin{aligned}
L_{1,n}\big(u_n(x)\big) &= -(1/2)\Gamma_3(\widetilde{\zeta}_n)(1/x) + (3/2)\big(2\Gamma_4(\widetilde{\zeta}_n) - 3\Gamma_3^2(\widetilde{\zeta}_n)\big) \\
&+ (1/48)\big(72\Gamma_5(\widetilde{\zeta}_n) - 394\Gamma_3(\widetilde{\zeta}_n)\Gamma_4(\widetilde{\zeta}_n) \\
&+ 267\Gamma_3^3(\widetilde{\zeta}_n)\big)\, x + \dots
\end{aligned} \tag{1.23}$$

$$\begin{aligned}
L_{2,n}\big(u_n(x)\big) &= (1/24)\big(3\Gamma_4(\widetilde{\zeta}_n) - 5\Gamma_3^2(\widetilde{\zeta}_n)\big) \\
&+ (1/24)\big(3\Gamma_5(\widetilde{\zeta}_n) - 16\Gamma_3(\widetilde{\zeta}_n)\Gamma_4(\widetilde{\zeta}_n) \\
&+ 15\Gamma_3^3(\widetilde{\zeta}_n)\big)\, x + \dots
\end{aligned} \tag{1.24}$$

Theorem 2 *For the distribution function* $F_{\widetilde{\zeta}_n}(x)$ *of the r.v.* $\widetilde{\zeta}_n$ *defined by equality* (1.5) *in the interval*

$$0 \le x < \Delta_n^*,$$

the relations of large deviations

$$\frac{1 - F_{\widetilde{\zeta}_n}(x)}{1 - \Phi(x)} = \exp\big\{L_n(x)\big\}\Big(1+\theta_1 f(x)\frac{x+1}{\Delta_n^*}\Big),$$

$$\frac{F_{\widetilde{\zeta}_n}(-x)}{\Phi(-x)} = \exp\big\{L_n(-x)\big\}\Big(1+\theta_2 f(x)\frac{x+1}{\Delta_n^*}\Big) \tag{1.25}$$

are valid. Here

$$f(x) = 60\big(1+10\Delta_n^* \exp\big\{-(1-x/\Delta_n^*)\sqrt{\Delta_n^*}\big\}\big)\big(1-x/\Delta_n^*\big)^{-1} \tag{1.26}$$

and $L_n(x)$ *is determined by* (1.17).

Corollary 1 *For* $x \ge 0$, $x = o\big(\Delta_n^{1/3}\big)$ *as* $\Delta_n \to \infty$, *where* Δ_n *is determined by* (1.13)

$$\lim_{n\to\infty} \frac{1 - F_{\widetilde{\zeta}_n}(x)}{1 - \Phi(x)} = 1\,, \qquad \lim_{n\to\infty} \frac{F_{\widetilde{\zeta}_n}(-x)}{\Phi(-x)} = 1 \tag{1.27}$$

Theorem 3 *For the r.v.* $\widetilde{\zeta}_n$ *defined by equality* (1.5)

$$\mathbf{P}\big\{\pm\widetilde{\zeta}_n \ge x\big\} \le \begin{cases} \exp\big\{-\frac{1}{4}x^2\big\}, & 0 \le x \le \Delta_n, \\ \exp\big\{-\frac{1}{4}\Delta_n x\big\}, & x \ge \Delta_n, \end{cases} \tag{1.28}$$

where Δ_n *is determined by* (1.13).

Next, let

$$L_{3,n} := B_n^{-3}\sum_{j=1}^{n} \mathbf{E}|Z_j - \mathbf{E}Z_j|^3\,, \quad L_n^{-1} = 64 B_n^{-3}\sum_{j=1}^{n}|\mu_j|^3 \tag{1.29}$$

Considering that $\mathbf{E}Y_j^6 = 15$ and $|a+b|^3 \le 4(|a|^3 + |b|^3)$, we obtain

$$L_{3,n} \le L_n^{-1} \le 16\Delta_n^{-1}\,, \tag{1.30}$$

where Δ_n is defined by equality (1.13).

Theorem 4 *For the distribution function* $F_{\widetilde{\zeta}_n}(x)$ *of the r.v.* $\widetilde{\zeta}_n$ *determined by equality* (1.5) *the inequality*

$$\sup_x \left|F_{\widetilde{\zeta}_n}(x) - \Phi(x)\right| \leq \frac{62.8}{\sqrt{2\pi}L_n} \tag{1.31}$$

holds. The inequality

$$\sup_x \left|F_{\widetilde{\zeta}_n}(x) - \Phi(x)\right| \leq 18/\Delta_n^* \tag{1.32}$$

is also true, where Δ_n^* *is determined by* (1.15)

Theorem 5 *For the distribution density* $p_{\widetilde{\zeta}_n}(x)$ *of the r.v.* $\widetilde{\zeta}_n$ *defined by equality* (1.5) *in the interval*

$$0 \leq x < \Delta_n^*,$$

for integer l, $l \geq 1$, *the equality*

$$\begin{aligned} \frac{p_{\widetilde{\zeta}_n}(x)}{\varphi(x)} &= \exp\left\{L_n(x)\right\}\Bigg(1 + \sum_{\nu=0}^{l-1} M_{\nu,n}(x) + \theta_1 q(l)\Big(\frac{x+1}{\Delta_n^*}\Big)^l \\ &+ \theta_2 \frac{2\pi e^2}{3} \frac{B_n}{\prod\limits_{k=1}^{4} |\mu_{i_k}|^{1/4}} \exp\Big\{-\frac{1}{5}T_n^2\Big\}\Bigg) \end{aligned} \tag{1.33}$$

holds. For polynomials $M_{\nu,n}(x)$ *the following formula holds:*

$$M_{\nu,n}(x) = \sum_{k=0}^{\nu} K_{k,n}(x) q_{\nu-k,n}(x), \tag{1.34}$$

where

$$\begin{aligned} K_{\nu,n}(x) &= \sum \prod_{m=1}^{\nu} \frac{1}{k_m!}\Big(-\lambda_{m+2,n}x^{m+2}\Big)^{k_m}, \\ K_0(x) &\equiv 1, \\ q_{\nu,n}(x) &= \sum H_{\nu+2l}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!}\left(\frac{\Gamma_{m+2}(\widetilde{\zeta}_n)}{(m+2)!}\right)^{k_m}, \end{aligned}$$

$q_{0,n}(x) \equiv 1$, $H_l(x)$ *are Chebyshev-Hermite polynomials, and the summation is taken over all integer solutions of the equation* $k_1 + 2k_2 + \ldots + \nu k_\nu = \nu$.

In a special case,

$$
\begin{aligned}
M_{0,n}(x) &\equiv 0,\\
M_{1,n}(x) &= -(1/2)\Gamma_3(\widetilde{\zeta}_n)\,x,\\
M_{2,n}(x) &= (1/8)\big(5\Gamma_3^2(\widetilde{\zeta}_n)-2\Gamma_4(\widetilde{\zeta}_n)\big)\,x^2+(1/24)\big(3\Gamma_4(\widetilde{\zeta}_n)-5\Gamma_3^2(\widetilde{\zeta}_n)\big),\\
M_{3,n}(x) &= (1/48)\big(34\Gamma_3(\widetilde{\zeta}_n)\Gamma_4(\widetilde{\zeta}_n)-4\Gamma_5(\widetilde{\zeta}_n)-45\Gamma_3^3(\widetilde{\zeta}_n)\big)x^3\\
&+\ (1/48)\big(6\Gamma_5(\widetilde{\zeta}_n)-35\Gamma_3(\widetilde{\zeta}_n)\Gamma_4(\widetilde{\zeta}_n)+35\Gamma_3^3(\widetilde{\zeta}_n)\big)x,\ldots.
\end{aligned}
$$

We get the expression of the quantity $q(l)$ from (6.11) [1], supposing that $\gamma=0$:

$$
q(l)=\big(3\sqrt{2e}/2\big)^l 8(l+2)^2 4^{3(l+1)}\Gamma\big((3l+1)/2\big). \tag{1.35}
$$

The quantities B_n, T_n and the function $\varphi(x)$ are defined by equalities (1.4), (1.15) and (1.6), respectively.

Theorem 6 *For the distribution density function* $p_{\widetilde{\zeta}_n}(x)$ *of the r.v.* $\widetilde{\zeta}_n$, *defined by* (1.5)

$$
\begin{aligned}
\sup_x\big|p_{\widetilde{\zeta}_n}(x)-\varphi(x)\big| &\le \frac{72}{\pi L_n}+\frac{1}{\sqrt{\pi}}\exp\Big\{-\frac{1}{64}L_n^2\Big\}\\
&+\ \frac{e^2B_n}{4\prod_{k=1}^4|\mu_{i_k}|^{\frac14}}\exp\Big\{-\frac14\Big(\frac{\Delta_n}{64}\Big)^2\Big\},
\end{aligned}
$$

where $i_k=\overline{1,n}$, $k=1,2,3,4$ *and the quantities* L_n, Δ_n *are defined by equalities* (1.29), (1.13), *respectively. Besides,* $L_n\ge\Delta_n/16$.

2 Proof of Theorems

2.1 Proof of Theorem 5

Since for the k^{th} order cumulant $\Gamma_k(\widetilde{\zeta}_n)$, $k=2,3,\ldots$, of the r.v. $\widetilde{\zeta}_n$, estimate (1.12) holds, for the r.v. $\xi=\widetilde{\zeta}_n$ the condition (S_γ) with $\gamma=0$ and $\Delta=\Delta_n$, Δ_n being defined by equality (1.13) of Lemma (6.1) [1] is satisfied. Basing on this lemma we have to estimate the integral

$$
R_n=\int\limits_{|t|\ge T_n}\big|f_{\widetilde{\eta}_n(h)}(t)\big|\,dt, \tag{2.1}
$$

where the quantity T_n is defined by equality (1.15), and

$$\widetilde{\eta}_n(h) = \big(\eta_n(h) - M_n(h)\big)/B_n(h), \quad \eta_n(h) = \sum_{j=1}^{n} Z_j(h), \tag{2.2}$$

where $Z_j(h)$ is conjugate $Z_j := \mu_j Y_j^2$, $j = 1, 2, \ldots, n$, r.v. with the density function

$$p_{Z_j(h)}(x) = e^{hx} p_{Z_j}(x) \Big(\int_{-\infty}^{\infty} e^{hx} p_{Z_j}(x)\, dx \Big)^{-1} \tag{2.3}$$

and

$$M_n(h) = \mathbf{E}\eta_n(h), \quad B_n^2(h) = \mathbf{D}\eta_n(h), \quad f_{\widetilde{\eta}_n(h)}(t) = \mathbf{E}\exp\big\{it\widetilde{\eta}_n(h)\big\}.$$

Further, let

$$\varphi_{Z_j}(h) := \mathbf{E}e^{hZ_j} = \int_{-\infty}^{\infty} e^{hx} p_{Z_j}(x)\, dx. \tag{2.4}$$

Since $f_{Z_j}(t) = \mathbf{E}\exp\{itZ_j\} = \varphi_{Z_j}(it)$, taking into account the expression of $f_{Z_j}(t)$ by equality (1.8), we obtain

$$\varphi_{Z_j}(h) = \big(1 - 2\mu_j h\big)^{-1/2}, \quad j = 1, 2, \ldots, n. \tag{2.5}$$

Hence, basing on the expression of the density $p_{Z_j(h)}(x)$ of the r.v. $Z_j(h)$ by equality (2.3), we get

$$f_{Z_j(h)}(t) = \frac{\varphi_{Z_j}(h + it)}{\varphi_{Z_j}(h)} = \big(1 - 2\nu_j(h)it\big)^{-1/2}, \tag{2.6}$$

where

$$\nu_j(h) = \mu_j/(1 - 2\mu_j h), \quad j = 1, 2, \ldots, n. \tag{2.7}$$

Recalling that Y_j, $j = 1, 2, \ldots, n$, are independent (0,1)– Gaussian r.v's, we obtain

$$f_{\widetilde{\eta}_n(h)}(t) = \exp\Big\{-it\frac{M_n(h)}{B_n(h)}\Big\} \prod_{j=1}^{n} f_{Z_j(h)}\Big(\frac{t}{B_n(h)}\Big). \tag{2.8}$$

Note that

$$M_n(h) = \mathbf{E}\eta_n(h) = \sum_{j=1}^{n} \nu_j(h), \quad B_n^2(h) = \mathbf{D}\eta_n(h) = 2\sum_{j=1}^{n} \nu_j^2(h),$$

where $\nu_j(h)$ is defined by equality (2.7).

From this, basing on equality (2.8), we derive

$$\left|f_{\widetilde{\eta}_n(h)}(t)\right| = \prod_{j=1}^{n} \Big(1+\frac{4\nu_j^2(h)}{B_n^2(h)}t^2\Big)^{-1/4}. \tag{2.9}$$

Recalling that the r.v. $\eta_n = \sum_{j=1}^{n} Z_j$, where $Z_j := \mu_j Y_j^2$, $j = 1, 2, \ldots, n$, are independent r.v's, we get

$$\varphi_{\eta_n}(h) = \mathbf{E}e^{h\eta_n} = \exp\Big\{\sum_{k=2}^{\infty} \frac{1}{k!}\Gamma_k(\eta_n)h^k\Big\}. \tag{2.10}$$

Then the mean $M_n(h)$ and variance $B_n^2(h)$ of the r.v. $\eta_n(h)$ defined by equality (2.3) are equal to :

$$\begin{aligned} M_n(h) &= \frac{d}{dh}\ln\varphi_{\eta_n}(h) = \\ &= \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\Gamma_k(\eta_n)h^{k-1}, \\ B_n^2(h) &= \frac{d^2}{dh^2}\ln\varphi_{\eta_n}(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!}\Gamma_k(\eta_n)h^{k-2}, \end{aligned} \tag{2.11}$$

respectively. Hence, basing on the expression of $\Gamma_k(\eta_n)$, by equality (1.10) we obtain

$$\begin{aligned} B_n^2(h) &= B_n^2\Big(1+\theta\sum_{k=3}^{\infty}(k-1)\big(2\max_{1\le j\le n}|\mu_j|h\big)^{k-2}\Big) \\ &= B_n^2\big(1+\theta(1/5)\big) \end{aligned} \tag{2.12}$$

for all $0 \le h \le \Delta_n/(12B_n)$, where B_n and Δ_n are defined by equalities (1.4) and (1.13), respectively. Now, recalling the definition of $\nu_j(h)$, by equality (2.7) and the fact that $0 \le h \le (1/12)\big(2\max_{1\le j\le n}|\mu_j|\big)^{-1}$ we get

$$\nu_j(h) = \mu_j(1-2\mu_j h)^{-1} = \mu_j\big(1+\theta(1/11)\big), \quad j = 1, 2, \ldots, n. \tag{2.13}$$

Next, using equalities (2.1) and (2.9), we have

$$R_n = \int\limits_{|t|\geq T_n} exp\Big\{-\frac{1}{4}\sum_{\substack{j=1\\ j\neq i_k}}^{n} \ln\Big(1+\frac{4\nu_j^2(h)}{B_n^2(h)}t^2\Big)\Big\}\prod_{k=1}^{4}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big|\,dt. \tag{2.14}$$

It is easy to check that

$$\prod_{k=1}^{2}\Big(1+\frac{4\nu_{i_k}^2(h)}{B_n^2(h)}t^2\Big) \geq \Big(1+\frac{4|\nu_{i_1}(h)\nu_{i_2}(h)|}{B_n^2(h)}t^2\Big)^2.$$

Consequently,

$$\prod_{k=1}^{2}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big| \leq \Big(1+\frac{4|\nu_{i_1}(h)\nu_{i_2}(h)|}{B_n^2(h)}t^2\Big)^{-1/2}. \tag{2.15}$$

Then

$$\int_{-\infty}^{\infty}\prod_{k=1}^{2}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big|\,dt \leq \frac{\pi}{2}\Big(\frac{B_n^2(h)}{|\nu_{i_1}(h)\nu_{i_2}(h)|}\Big)^{1/2}. \tag{2.16}$$

Hence, making use of the Cauchy-Schwarz inequality, we obtain

$$\int_{-\infty}^{\infty}\prod_{k=1}^{4}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big|\,dt \leq \frac{\pi}{2}\frac{B_n(h)}{\prod_{k=1}^{4}|\nu_{i_k}(h)|^{1/4}}. \tag{2.17}$$

Now, making use of equalities (2.12) and (2.13), one can easily check that $0 < 4\nu_j^2(h)T_n^2/B_n^2(h) < 1$. Thus, basing on the inequality $\ln(1+x) > x/2$, $0 < x < 1$, we have

$$\ln\Big(1+\frac{4\nu_j^2(h)}{B_n^2(h)}T_n^2\Big) \geq \frac{8\mu_j^2}{5B_n^2}T_n^2. \tag{2.18}$$

Hence, taking into account equalities (2.14) and (2.17), we obtain the estimate of integral R_n:

$$R_n \leq \frac{2\pi e^2}{3}\frac{B_n}{\prod_{k=1}^{4}|\mu_{i_k}|^{1/4}}\exp\Big\{-\frac{1}{5}T_n^2\Big\}, \tag{2.19}$$

where T_n is defined by equality (1.15).

2.2 Proof of Theorem 1

Since for the k^{th} order cumulant $\Gamma_k(\widetilde{\zeta}_n)$, $k = 2, 3, \dots$, of the r. v. $\widetilde{\zeta}_n$, estimate (1.12) holds, for the r.v. $\xi = \widetilde{\zeta}_n$, the condidion (S_γ) with $\gamma = 0$ and $\Delta = \Delta_n$, Δ_n being defined by equality (1.13) of Lemma 1 [3] is satisfied. Basing on this lemma, we have to estimate the integral

$$R_n^* = \int\limits_{T_n}^{c\Delta_n^{l-2}} \left|f_{\widetilde{\eta}_n(h)}(t)\right| \frac{dt}{t},$$

where the quantity T_n and the function $f_{\widetilde{\eta}_n(h)}(t)$ are defined by equalities (1.15) and (2.8), respectively. Considering that $R_n^* \le (1/T_n)R_n$, where the integral R_n is defined by equality (2.1), and making use estimate (2.19), we obtain

$$R_n^* \le \frac{72\sqrt{2}\pi e^2}{\left(1 - x/\Delta_n^*\right)} \frac{2 \max\limits_{1 \le j \le n} |\mu_j|}{\prod_{k=1}^4 |\mu_{i_k}|^{1/4}} \exp\left\{-\frac{1}{5} T_n^2\right\}. \tag{2.20}$$

Basing on Lemma 1 [3] and estimates (1.12),(2.20), we obtain the assertion of Theorem 1.

To prove Theorem 2, we have to use Lemma (2.3) in [1] for the r.v. $\xi = \widetilde{\zeta}_n$, for the k^{th} order cumulant $\Gamma_k(\widetilde{\zeta}_n)$, $k = 2, 3, \dots$, of which estimate (1.13) holds. We complete the proof of this theorem making use of equalities (1.17),(1.18) and (1.25),(1.26).

To prove Theorem 3, we have to make use of Lemma 2.4 in [1] for the r.v. $\xi = \widetilde{\zeta}_n$, for the order k^{th} cumulant $\Gamma_k(\widetilde{\zeta}_n)$, $k = 2, 3, \dots$, of which estimate (1.13) is valid, considering that $(k-1)! \le (1/2)k!$, $k = 2, 3, \dots$.

2.3 Proof of Theorem 6

Let a r.v. $\widetilde{\eta}_n := B_n^{-1}(\eta_n - \mathbf{E}\eta_n)$. Then its characteristic function

$$f_{\widetilde{\eta}_n}(t) = \exp\left\{-itB_n^{-1} \sum_{j=1}^n \mu_j\right\} \prod_{j=1}^n \left(1 - 2\mu_j B_n^{-1} it\right)^{-1/2}. \tag{2.21}$$

In a view of the fact that the characteristic function $f_{\widetilde{\zeta}_n}(t) = f_{\widetilde{\eta}_n}(t)$

of the r.v. $\widetilde{\zeta}_n$ defined by equality (1.5), we have

$$\begin{aligned}\sup_x \big| p_{\widetilde{\zeta}_n}(x) - \varphi(x) \big| &\le \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \big| f_{\widetilde{\eta}_n}(t) - \exp\big\{ -\frac{1}{2}\, t^2 \big\} \big| dt \\ &= \frac{1}{2\pi} \big(I_1 + I_2^{(1)} + I_2^{(2)} \big),\end{aligned}$$

where

$$I_1 = \int\limits_{|t| \le (1/4)L_n} \big| f_{\widetilde{\eta}_n}(t) - \exp\big\{ -\frac{1}{2}\, t^2 \big\} \big| dt, \tag{2.22}$$

$$I_2^{(1)} = \int\limits_{|t| \ge (1/4)L_n} \big| f_{\widetilde{\eta}_n}(t) \big| dt, \quad I_2^{(2)} = \int\limits_{|t| \ge (1/4)L_n} \exp\big\{ -\frac{1}{2}\, t^2 \big\} dt, \tag{2.23}$$

and the quantity L_n is determined by equality (1.29). Recalling the definition of the quantity L_n by equality (1.29), and basing on Lemma 1([6] p.155) and inequality (1.30), we get

$$\big| f_{\widetilde{\zeta}_n}(t) - \exp\{ -\frac{1}{2}\, t^2 \} \big| \le \frac{16}{L_n} |t|^3 \exp\{ -\frac{1}{3}\, t^2 \}, \tag{2.24}$$

in the interval $|t| \le (1/4)L_n$. Basing on this inequality, we get the estimate $I_1 \le 144/L_n$ of integral I_1 defined by equality (2.22). It is easy to see that for integral $I_2^{(2)}$, defined by equality (2.23), the estimate $I_2^{(2)} \le 2\sqrt{\pi} \exp\{-(1/64)L_n^2\}$ is valid. It remained to estimate the integral $I_2^{(1)}$ that is defined by equality (2.23). The shortest way to do that is to make use of the estimate of integral R_n defined by equality (2.1), considering that

$$\begin{aligned} f_{\widetilde{\eta}_n}(t) &= f_{\widetilde{\eta}_n(h)}(t) \big|_{h=0},\ B_n(h) \big|_{h=0} \\ &= B_n,\ \nu_j(h) \big|_{h=0} = \mu_j,\ j = 1, 2, \ldots, n. \end{aligned} \tag{2.25}$$

Now, basing on equality (2.14), we have

$$\begin{aligned} I_2^{(1)} &= \int\limits_{|t| \ge (1/4)L_n} \exp\Big\{ -\frac{1}{4} \sum_{\substack{j=1 \\ j \ne i_k}}^{n} \ln\big(1 + (4\mu_j^2/B_n^2)t^2\big) \Big\} \times \\ &\times \prod_{k=1}^{4} \big| f_{Z_{i_k}}\big(t/B_n\big) \big|\, dt. \end{aligned} \tag{2.26}$$

Next, making use of the Cauchy-Schwarcz inequality, and of the inequality (2.16), we obtain

$$\int_{-\infty}^{\infty} \prod_{k=1}^{4} \left| f_{Z_{i_k}}(t/B_n) \right| dt \leq \frac{\pi}{2} \frac{B_n}{\prod_{k=1}^{4} |\mu_{i_k}|^{1/4}}. \tag{2.27}$$

In turn, since $(1/4)L_n \geq (1/64)\Delta_n$, we have $|t| \geq (1/4)L_n$

$$\ln\Big(1+\frac{4\mu_j^2}{B_n^2}t^2\Big) \geq \frac{1}{2} \frac{\mu_j^2}{(64 \max\limits_{1\leq j\leq n} |\mu_j|)^2}. \tag{2.28}$$

So, basing on inequalities (2.27), (2.28) and expression (2.26) of integral $I_2^{(1)}$, we obtain

$$I_2^{(1)} \leq \frac{\pi e^2}{2} \frac{B_n}{\prod_{k=1}^{4} |\mu_{i_k}|^{1/4}} \exp\Big\{ -\frac{1}{4}\Big(\frac{\Delta_n}{64}\Big)^2 \Big\}.$$

Finally, basing on this and the obtained estimates of integrals I_1 and $I_2^{(2)}$, we conclude the assertion of the theorem.

2.4 Proof of Theorem 4

We derive inequality (1.32) by applying the conclusion (2.1) in [1] for the r.v. $\xi = \widetilde{\zeta}_n$ to the k^{th} order cumulant $\Gamma_k(\widetilde{\zeta}_n)$, $k = 2, 3, \dots,$ of which estimate (1.12) holds.

To prove inequality (1.31), we make use of V.M.Zolotarev's Lemma 1 [2]. According to this lemma (or using inequality (14) [5]), we obtain

$$\begin{aligned}\sup_x \left| F_{\widetilde{\zeta}_n}(x) - \Phi(x) \right| &\leq \frac{17.4}{\sqrt{2\pi} L_n} + \frac{3}{2\pi} \int_0^{(1/4)L_n} \left| f_{\widetilde{\eta}_n}(t) - \exp\Big\{ -\frac{1}{2}t^2 \Big\} \right| \frac{dt}{t} \\ &\leq \frac{17.4}{\sqrt{2\pi} L_n} + \frac{44.4}{\sqrt{2\pi} L_n} = \frac{62.8}{\sqrt{2\pi} L_n},\end{aligned}$$

where the quantity L_n is defined by equality (1.29), and this is the proposition of Theorem 4.

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