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Linearly Invariant Classes of Functions Analytical in the Half-Plane

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Abstract. In present paper the properties of classes of introduced by authors functions analytical in the half-plane are investigated. These classes constructed by using special normalization of functions, automorphism of the half-plane and by operator actual on these classes. Such classes are called as linearly invariant ones. In case when domain is unit disc similar linearly invariant classes were considered by Ch. Pommerenke, V. Starkov, E.G. Kiriyatzkii.

Keywords: operator, analytical functions, linearly invariant classes, half-plane, ogrand.

1 Major notational conventions, definitions and auxiliary statements

Let Π is a half-plane $\operatorname{Re} z > 0$, $A_n(\Pi)$ – class of analytical in Π functions $F(z)$ with condition $F^{(n)}(z) \neq 0$, $\forall z \in \Pi$, $\widetilde{A}_n(\Pi)$ – class of analytical in Π functions $F(z)$ from $A_n(\Pi)$, which are normalized by conditions:

$$F(1) = F'(1) = \ldots = F^{(n-1)}(1) = 0, \quad F^{(n)}(1) = n!\,.$$

It is obvious that for any fixed $m \geq 2$ every function $F(z)$ of $\widetilde{A}_n(\Pi)$ can be represented in form

$$F(z) = (z-1)^n + \sum_{k=2}^{m} a_{k,n}(z-1)^{n+k-1} + \Psi_m(z),$$

where $\Psi_m(z)$ – dependent on $F(z)$ analytical in Π function. Number

$$a_{k,n} = \frac{F^{(n+k-1)}(1)}{(n+k-1)!}$$

we call by k-th coefficient of function $F(z)$. Let us introduce the operator

$$N_n[F] = \frac{n!}{F^{(n)}(1)} \left(F(z) - \sum_{k=0}^{n-1} \frac{F^{(k)}(1)}{k!}(z-1)^k \right),$$

which we call by *normalizing operator*. This operator transfers any function from $A_n(\Pi)$ to a function of class $\widetilde{A}_n(\Pi)$. For $n = 0$ we set that $N_0[F] = F(z)/F(1)$. Note that $N_n[cF + P] = N_n[F]$, where $c \neq 0$ and P, is a polynomial of the degree no higher than $n - 1$ and that $N_n\big[N_n[F]\big] = N_n[F]$.

Denote by $A(D)$ class of analytical in domain D functions. The n-th order divided difference of function $F(z) \in A(D)$ define (see [1, 2]) by formula

$$\big[F(z); z_0, \ldots, z_n\big] = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{F(\xi)d\xi}{(\xi - z_0)\ldots(\xi - z_n)},$$

where Γ is a simple closed contour, located in D and covering all the points $z_0, \ldots, z_n \in D$. In above formula among the points $z_0, \ldots, z_n \in D$ may occur coincident. Denote by L a set of functions of shape $w = tz$, where $t > 0$. Let us arbitrarily choose $w \in L$ and introduce omega-operator of n-th order by formula

$$\Omega_n^w[F] = \frac{(z-1)^n \big[F(z); w(z), \overbrace{t, \ldots, t}^{n}\,\big]}{\dfrac{1}{n!}F^{(n)}(t)}.$$

This operator for any fixed $w = tz$ is defined on class $A_n(\Pi)$ and transfers every function of class $A_n(\Pi)$ to the function of class $\widetilde{A}_n(\Pi)$. Let us introduce the operators

$$\mu_{2,n}[F] = \frac{zF^{(n+1)}(z)}{(n+1)F^{(n)}(z)}, \quad F(z) \in \widetilde{A}_n(\Pi),$$
$$\nu_n[F] = \ln \frac{F^{(n)}(z)}{n!}, \qquad F(z) \in \widetilde{A}_n(\Pi).$$

Note, that if one fix some function of $F(z) \in \widetilde{A}_n(\Pi)$, then above operators became a functions of $z \in \Pi$. If we will fix $z \in \Pi$, then these operators convert to functionals, defined on class $\widetilde{A}_n(\Pi)$. The below lemma is nearly clear.

Lemma 1. *The equations*

$$\mu_{2,n}\big[F(z)\big] = \frac{z}{n+1}\,\frac{d}{dz}\,\nu_n\big[F(z)\big], \quad \nu_n\big[F(z)\big] = \int\limits_1^z \frac{n+1}{z}\mu_{2,n}\big[F(z)\big]dz$$

holds true.

2 Some properties of operator $\Omega_n^w,\ w \in L$

In this paragraph we remind some properties of operator Ω_n^w introduced in paper [3] and give some new properties of this operator.

Theorem 1. *If $F(z) \in \widetilde{A}(\Pi)$ and $w = tz \in L$, then*

$$\Omega_n^w\big[F(z)\big] = N_n\big[F(w)\big],$$

where

$$N_n\big[F(w)\big] = \frac{n!}{F^{(n)}(t)t^n}\big(F(tz) - P(z;t)\big), \quad P(z;t) = \sum_{k=0}^{n-1} \frac{F^{(k)}(t)}{k!}t^k(z-1)^k.$$

Let

$$\Phi_{n,a}(z) = (z-1)^n + \sum_{k=2}^{\infty} c_{k,n}(z-1)^{n+k-1},$$

where

$$c_{n,k} = \frac{n!}{(n+k-1)!}(n+1)a\big((n+1)a - 1\big)\dots\big((n+1)a - (k-2)\big)$$

for $k = 2, 3, \dots$. Note that $c_{2,n} = a$. Function $\Phi_{n,a}(z)$ we call by *main* one.

Theorem 2. *The main function belongs to class $\widetilde{A}_n(\Pi)$ and is a fixed function of operator Ω_n^w, i.e.*

$$\Omega_n^w[\Phi_{n,a}] = \Phi_{n,a}, \quad \forall w \in L.$$

Let k-th coefficient of some function $F(z) \in \widetilde{A}_n(\Pi)$ is equal to number b_k, where $k \geq 2$. If b_k is the k-th coefficient of function $F(z;t) = \Omega_n^w\big[F(z)\big]$ for any $w \in L$, then k-th coefficient of function $F(z)$ we will call by *invariant coefficient* of this function.

Theorem 3. *Let equation*

$$\frac{n!}{(n+k-1)!}\prod_{m=0}^{k-2}\big((n+1)a-m\big)=b_k$$

with respect to a has $k-1$ of pairwise different roots $a_1,\dots,a_{k-1}$. Then only functions of form

$$F(z)=\sum_{m=1}^{k-1}c_m\Phi_{n,a_m}(z),\quad c_1+\ldots+c_{k-1}=1$$

has number b_k as theirs k-th invariant coefficient.

Set of transformations $w=tz\in L$ is a group, if we define product of two transformations w_1 and w_2 by formula $w_3=w_1(w_2)$.

Theorem 4. *Let $w_1,\dots,w_k\in L$ and $F_2=\Omega_n^{w_1}[F_1]$, $F_3=\Omega_n^{w_2}[F_2]$, $F_{k+1}=\Omega_n^{w_k}[F_k]$. Then $F_{k+1}=\Omega_n^{w_{k+1}}[F_1]$, where $w_{k+1}=w_1\big(w_2(\ldots(w_k)\ldots)\big)$.*

Proof. It is enough to prove the theorem for $k=2$. Using properties of normalizing operator we get

$$\begin{aligned}F_3(z)&=\Omega_n^{w_2}\big[F_2(z)\big]=\Omega_n^{w_2}\Big[\Omega_n^{w_1}\big[F_1(z)\big]\Big]\\&=\Omega_n^{w_2}\Big[N_n\Big[F_1\big(w_1(z)\big)\Big]\Big]=N_n\Big[N_n\Big[F_1\Big(w_1\big(w_2(z)\big)\Big)\Big]\Big]\\&=N_n\Big[F_1\Big(w_1\big(w_2(z)\big)\Big)\Big]=N_n\Big[F_1\big(w_3(z)\big)\Big]=\Omega_n^{w_3}\big[F_1(z)\big].\end{aligned}$$ □

Theorem 5. *If $F_1\in\widetilde{A}_n(\Pi)$ and $F_2\in\widetilde{A}_n(\Pi)$, where $F_1\neq F_2$, then*

$$\Omega_n^w[F_1]\neq\Omega_n^w[F_2],\quad\forall w\in L.$$

Proof. Let, contrary to the statement of theorem, for some $w=tz\in L$ the identity $\Omega_n^w\big[F_1(z)\big]\equiv\Omega_n^w\big[F_2(z)\big]$ takes place. Using Theorem 1 we get identity $F_1(z)=\varphi(t)F_2(z)+R(z;t)$, where $\varphi(t)\neq0$ and $R(z;t)$ is a polynomial of the degree no higher than $n-1$. Functions $F_1(z)$ and $F_2(z)$ are normalized in the half-plane Π, so $\varphi(t)=1$ and $R(z;t)\equiv0$. Therefore $F_1=F_2$. Obtained contradiction proves the theorem. □

3 Definition of linearly invariant class

Set S of functions $F(z)$ of $\widetilde{A}_n(\Pi)$ we will call by linearly invariant class of n-th order, if from belonging $F(z) \in S$ follows $\Omega_n^w\big[F(z)\big] \in S$ for any $w \in L$.

Let us give some examples of linearly invariant classes of n-th order.

Example 1. $\widetilde{A}_n(\Pi)$ is a linear invariant class. In fact, operator Ω_n^w transfers function $F(z)$ from $\widetilde{A}_n(\Pi)$ to function of the same class $\widetilde{A}_n(\Pi)$, for any $w \in L$. Note also that $\widetilde{A}_n(\Pi)$ contains any of linearly invariant classes.

Example 2. Let us fix in $\widetilde{A}_n(\Pi)$ function $F(z)$ and make up the class of functions $\Psi_w(z) = \Omega_n^w\big[F(z)\big]$, where w vary over all set L. Due to Theorem 4 (on chain), such class must be linearly invariant one. We will call this class as *simple* linearly invariant class and denote it by $\widetilde{\mathfrak{R}}_n(\Pi; F)$. Function $F(z)$ we will call by generator of simple class. For simple class we have the following

Property 1. *If $F_1(z) \in \widetilde{\mathfrak{R}}_n(\Pi; F)$, then $F(z) \in \widetilde{\mathfrak{R}}_n(\Pi; F_1)$ for any $w \in L$. In other words, if function $F(z)$ is the generator of simple class and $F_1(z) \in \widetilde{\mathfrak{R}}_n(\Pi; F)$, then function $F_1(z)$ must be the generator of this simple class too.*

Validity of this statement follows from construction of simple family together with Theorem 4 (on chain). The following properties of simple family are near clear.

Property 2. *If pair of simple classes has a common function, then they are fully coincident.*

Property 3. *Union of the simple classes is a linearly invariant class.*

Theorem 6. *Let function F runs over whole set of functions from simple class $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$. Then function $\Psi = \Omega_n^w[F]$ for any fixed $w \in L$ runs over whole set of functions from $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$ too.*

Proof. Let $w \in L$ is arbitrarily fixed. Since $F \in \widetilde{\mathfrak{R}}_n(\Pi; F_0)$, then Property 1 implies that F (as well as F_0) is the generator of class $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$. Therefore $\Psi = \Omega_n^w F \in \widetilde{\mathfrak{R}}_n(\Pi; F_0)$. According to Theorem 5, two distinct functions of $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$ are reflected to pair of distinct functions of the same class. Let Ψ_0 is a arbitrary fixed function of $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$. Generate function $F^* = \Omega_n^{w^*}[\Psi_0]$, where

$w^*(w) = w_0 \equiv z$. As far as $\Psi_0 \in \widetilde{\mathfrak{R}}_n(\Pi; F_0)$, then keeping in mind Property 1 we have that function Ψ_0 (as well as F_0) is a generator of class $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$. Hence $F^* \in \widetilde{\mathfrak{R}}_n(\Pi; F_0)$. Further, it is easily seen that $\Psi_0 = \Omega_n^w\lfloor F^*\rfloor$. By Theorem 5 we get, that in class $\widetilde{\mathfrak{R}}_n(\Pi; F_0)$ there is one and only one function F^* corresponding to function $\Psi_0 \in \widetilde{\mathfrak{R}}_n(\Pi; F_0)$. So the proof of the Theorem 6 is complete. □

Example 3. Simple linearly invariant class generated by main function $\Phi_{n,a}(z)$ consists only of one function.

This statement follows from Theorem 4.

Union of a set of linearly invariant classes of n-th order denote by $\widetilde{\mathfrak{F}}_n(\Pi)$. Number

$$\delta_F = \sup_{z\in\Pi} \bigl|\mu_{2,n}[F]\bigr|$$

we call by *ogrand* of function $F(z)$. Number

$$\delta = \sup_{F\in\widetilde{\mathfrak{F}}(\Pi)} \delta_F$$

we call by ogrand of linearly invariant class $\widetilde{\mathfrak{F}}_n(\Pi)$. Simple class with ogrand δ we denote by $\widetilde{\mathfrak{R}}_n(\Pi; \delta)$. By using Theorem 6 one can establish validity of

Theorem 7. *The ogrand of some function of simple class coincides with the ogrand of simple class containing this function. In other words, the ogrand takes up constant value on the set of functions of simple class.*

Linearly invariant class with ogrand δ we denote by $\widetilde{\mathfrak{F}}_n(\Pi; \delta)$. Let us denote the union of all linearly invariant classes with ogrand not greater than δ by $\widetilde{U}_n(\Pi; \delta)$ and call it by *universal* class.

Denote by $K_n(D)$ class of analytic in domain D functions $F(z)$ such, that $\bigl[F(z); z_0, \dots, z_n\bigr] \neq 0$ for any set of pairwise distinct $z_0, \dots, z_n \in D$ (see [4]). For $n = 1$ one has, as it easily seen, class $K_1(D)$ of all univalent in D functions, which play a large role in conformal mapping theory and in geometrical theory of analytical functions [4, 5].

By using definition of class $K_n(\Pi)$ and elementary properties of divided differences [3], we get

Lemma 2. *If* $F(z) \in K_n(\Pi)$, *then*

$$cF(z) + P(z) \in K_n(\Pi),$$

where $c \neq 0$ *and* $P(z)$ *is a polynomial of the degree no higher than* $n-1$.

The following lemma is valid too [6]:

Lemma 3. *If* $F(z) \in K_n(\Pi)$, *then* $F^{(n)}(z) \neq 0$ *for any* $z \in \Pi$. *Conversely, if* $F^{(n)}(z) \neq 0$ *for all* $z \in \Pi$, *then for any point* $\xi \in \Pi$ *there exists its neighborhood (domain)* $O(\xi)$ *such, that* $F(z) \in K_n\big(O(\xi)\big)$, *i.e.* $F(z) \in K_n(\Pi)$ *locally in* Π.

Using Lemma 3 we arrive at conclusion, that in class $K_n(\Pi)$ one can evolve a subclass $\widetilde{K}_n(\Pi)$ of normalized functions. Lemma 3 shows that class $\widetilde{K}_n(\Pi)$ is the subclass of class $\widetilde{A}_n(\Pi)$. Furthermore, it is clear from Lemma 3, that if $F(z) \in \widetilde{A}_n(\Pi)$, then $F(z) \in K_n(\Pi)$ *locally* in Π.

Example 4. Class $\widetilde{K}_n(\Pi)$ – is a linearly invariant class.

In fact, let us fix arbitrarily transformation $w = tz \in L$ and points $z_0, \dots, z_n$. Let $w_k = tz_k,\ k = 0, 1, \dots, n$. By using properties of normalization operator and elementary properties of divided differences, we get

$$\frac{1}{n!} F^{(n)}(t) \big[\Omega_n^w \big[F(z)\big]; z_0, \dots, z_n\big] = \big[F(w); z_0, \dots, z_n\big]. \tag{1}$$

Since $F(z) \in \widetilde{K}_n(\Pi)$ and $w_0, \dots, w_n \in \Pi$, then right-hand member of (1) is nonzero. But then left-hand member of (1) is nonzero too. Taking into account the arbitrariness of $w = tz \in L$ and $z_0, \dots, z_n \in \Pi$, we get that $\Omega_n^w\big(F(z)\big) \in \widetilde{K}_n(\Pi)$. Thus, if $F(z) \in \widetilde{K}_n(\Pi)$, then $\Omega_n^w\big(F(z)\big) \in \widetilde{K}_n(\Pi)$ for any $w = tz \in L$. So, class $\widetilde{K}_n(\Pi)$ is a linearly invariant class. In case when $n = 1$ and domain is unit disc E linearly invariant classes were considered by Ch. Pommerenke [7] and by V. Starkov [8, 9].

4 Some tests of belonging to universal class

Theorem 8. $F(z) \in \widetilde{U}_n(\Pi; \delta)$ *if and only if*

$$\delta_F \leq \delta. \tag{2}$$

Proof. Necessity immediately follows from the definition of class $\widetilde{U}_n(\Pi;\delta)$ and from definition of ogrand δ. Let us establish sufficiency. Let function $F(z)$ satisfies the condition (2). By using this function generate simple class $\widetilde{\mathfrak{R}}_n(\Pi)$. Then, according to Theorem 7, ogrand of this class is equal to δ_F. Hence $\widetilde{\mathfrak{R}}_n(\Pi) \subset \widetilde{U}_n(\Pi;\delta)$ and consequently $F(z) \in \widetilde{U}_n(\Pi;\delta)$. $\square$

Corollary 1. *If $|a| \le \delta$, then $\Phi_{n,a}(z) \in \widetilde{U}_n(\Pi;\delta)$.*

In fact, the ogrand of function $\Phi_{n,a}(z)$ is equal to $|a|$.

Theorem 9. *If $F_m(z) \in \widetilde{U}_n(\Pi;\delta),\ m = 1,\dots,k$, then for any combination of positive $\lambda_m,\ m = 1,\dots,k$, such that $\lambda_1 + \ldots + \lambda_k = 1$, one has*

$$F(z) = \int\limits_1^z \cdots \int\limits_1^z \prod_{m=1}^{k} \left(F_m^{(n)}(z)\right)^{\lambda_m} dz \ldots dz \in \widetilde{U}_n(\Pi;\delta).$$

Proof. Note that

$$F^{(n)}(z) = \prod_{m=1}^{k} \left(F_m^{(n)}(z)\right), \quad F^{(n)}(z) \neq 0, \quad \forall z \in \Pi$$

and therefore $F(z) \in \widetilde{A}_n(\Pi)$. Further

$$\big|\mu_{2,n}\big[F(z)\big]\big| = \Big|\sum_{m=1}^{k} \lambda_m \mu_{2,n}\big[F(z)\big]\Big| \le \sum_{m=1}^{k} \lambda_m \big|\mu_{2,n}\big[F(z)\big]\big| \le \sum_{m=1}^{k} \lambda_m \delta = \delta.$$

Finally, from Theorem 7 it yields, that Theorem 8 is valid. $\square$

5 This section deals with estimations of modules

We need in lemma, which has a number of applications.

Lemma 4. *Let $u(x)$ be a continuous in a certain interval $[a,b]$ complex-valued function of real argument x. Equality*

$$\Big|\int\limits_a^b u(x)dx\Big| = \int\limits_a^b \big|u(x)\big|dx, \tag{3}$$

takes place if and only if all values of the function belong to segment of ray $l(\beta)$ which is outgoing from origin and inclined to real axis on the some angle β, i.e. if

$$u(x) = \big|u(x)\big|e^{i\beta}, \quad \forall x \in [a,b].$$

Proof. Let for mentioned function $u(x)$ the equality (3) takes place. Assume that

$$\int\limits_a^b u(x)dx = \Bigg|\int\limits_a^b u(x)dx\Bigg|e^{i\beta}.$$

Then equality (3) we can rewrite in the form

$$\int\limits_a^b \mathrm{Re}\big\{e^{-i\beta}u(x)\big\}dx = \int\limits_a^b \big|e^{-i\beta}u(x)\big|dx. \tag{4}$$

Further,

$$\mathrm{Re}\big\{e^{-i\beta}u(x)\big\} \le \big|e^{-i\beta}u(x)\big|, \quad \forall x \in [a,b]. \tag{5}$$

Functions $\mathrm{Re}\big\{e^{-i\beta}u(x)\big\}$ and $\big|e^{-i\beta}u(x)\big|$ are continuous in any point of segment $[a,b]$. Therefore from (4) and (5) it follows that

$$\mathrm{Re}\big\{e^{-i\beta}u(x)\big\} = \big|e^{-i\beta}u(x)\big|, \quad \forall x \in [a,b].$$

This implies that all values of function $u(x)$ belong to ray $l(\beta)$.

Let conversely all values of function $u(x)$ belong to ray $l(\beta)$. Then clear that

$$\Bigg|\int\limits_a^b u(x)dx\Bigg| = \Bigg|\int\limits_a^b \big|u(x)e^{i\beta}\big|dx\Bigg| = \int\limits_a^b \big|u(x)\big|dx. \qquad \square$$

Theorem 10. *For any function $F(z) \in \widetilde{A}_n(\Pi)$ the estimate*

$$\big|\nu_n\big[F(r)\big]\big| \le \big|\nu_n\big[\Phi_{n,\delta_F}(r)\big]\big|, \quad \forall r > 0 \tag{6}$$

holds true. Equality sign for $r = r_0 > 0$, $r_0 \neq 1$ realize only main functions $\Phi_{n,a}(z)$ from $\widetilde{A}_n(\Pi)$.

Proof. We have

$$\begin{aligned}\left|\nu_n\big[F(r)\big]\right| &= \left|\int\limits_1^r \frac{n+1}{r}\mu_{2,n}\big[F(r)\big]dr\right| \le \left|\int\limits_1^r \frac{n+1}{r}\left|\mu_{2,n}\big[F(r)\big]\right|dr\right| \\ &\le \left|\int\limits_1^r \frac{n+1}{r}\delta_F dr\right| = \left|\nu_n\big[\Phi_{n,\delta_F}(r)\big]\right|\end{aligned} \tag{7}$$

and inequality (6) is proved. Let for some $r = r_0 > 0$, $r_0 \neq 1$ in (6) the equality sign takes place, i.e.

$$\left|\nu_n\big[F(r_0)\big]\right| = \left|\nu_n\big[\Phi_{n,\delta_F}(r_0)\big]\right|. \tag{8}$$

Then from (7) and (8) we get

$$\left|\int\limits_1^{r_0} \frac{1}{r}\mu_{2,n}\big[F(r)\big]dr\right| = \left|\int\limits_1^{r_0} \frac{1}{r}\left|\mu_{2,n}\big[F(r)\big]\right|dr\right|, \tag{9}$$

$$\left|\int\limits_1^{r_0} \frac{1}{r}\left|\mu_{2,n}\big[F(r)\big]\right|dr\right| = \left|\int\limits_1^{r_0} \frac{1}{r}\delta_F dr\right|. \tag{10}$$

From (10) and taking into account properties of ogrand δ_F, we obtain, that

$$\left|\mu_{2,n}\big[F(r)\big]\right| = \delta_F \tag{11}$$

for any r between 1 and r_0. Applying Lemma 4 to equality (9) and taking into consideration (11) come to conclusion, that $\mu_{2,n}\big[F(r)\big] = \delta_F e^{i\beta}$ for any r between 1 and r_0. Since $\mu_{2,n}\big[F(z)\big]$ is an analytical in Π function, then, using principle of analytic continuation, we get

$$\mu_{2,n}\big[F(z)\big] = \delta_F e^{i\beta}, \quad \forall z \in \Pi.$$

Solving the last equation with respect to $F^{(n)}(z)$ obtain

$$F^{(n)}(z) = n! z^{(n+1)\delta_F e^{i\beta}} = \Phi_{n,a}^{(n)}(z), \tag{12}$$

where $a = \delta_F e^{i\beta}$ is the second coefficient of function $\Phi_{n,a}(z)$. This implies that function realizing equality sign in (6) is of shape $\Phi_{n,a}(z)$. Verify that function

$\Phi_{n,a}(z)$ in fact realizes equality sign in (6). For that let us substitute this function in (8). We have

$$\left| \ln \frac{\Phi^{(n)}_{n,a}(r_0)}{n!} \right| = \left| \ln \frac{\Phi^{(n)}_{n,|a|}(r_0)}{n!} \right|$$

or

$$\big| \ln r_0^{(n+1)a} \big| = \big| \ln r_0^{(n+1)|a|} \big|,$$

that is equivalent to evident equality

$$\big|(n+1)a\big| = \big|(n+1)|a|\big|.$$

So, we come to conclusion, that equality sign in (6) realizes only main functions of shape. $\square$

Corollary 2. *For any function* $F(z) \in \widetilde{U}_n(\Pi;\delta)$ *the estimate*

$$\big|\nu_n\big[F(r)\big]\big| \le \big|\nu_n\big[\Phi_{n,\delta}(r)\big]\big|, \quad \forall r > 0$$

holds true. Equality sign for $r = r_0 > 0$*,* $r_0 \neq 1$ *realize only main functions* $\Phi_{n,a}(z) \in \widetilde{U}_n(\Pi;\delta)$*, where* $|a| = \delta$*.*

Theorem 11. *If* $F(z) \in \widetilde{A}_n(\Pi)$*, then*

$$e^{-(n+1)\delta_F|\ln r|} \le \frac{\big|F^{(n)}(z)\big|}{n!} \le e^{(n+1)\delta_F|\ln r|}, \quad \forall r > 0. \tag{13}$$

Equality signs in (13) *for* $r = r_0 > 0$*,* $r_0 \neq 1$ *realize only main functions* $\Phi_{n,\pm a}(r) \in \widetilde{A}_n(\Pi)$*,* $a > 0$*.*

Proof. According to Theorem 9 we have

$$\big|\nu_n\big[F(r)\big]\big| \le \big|\nu_n\big[\Phi_{n,\delta_F}(r)\big]\big|, \quad \forall r > 0.$$

From here

$$-\big|\nu_n\big[\Phi_{n,\delta_F}(r)\big]\big| \le \operatorname{Re}\big\{\nu_n\big[F(r)\big]\big\} \le \big|\nu_n\big[\Phi_{n,\delta_F}(r)\big]\big|, \quad \forall r > 0$$

or

$$-\left|\ln\frac{\Phi^{(n)}_{n,\delta_F}(r)}{n!}\right| \le \mathrm{Re}\left\{\ln\frac{F^{(n)}(r)}{n!}\right\} \le \left|\ln\frac{\Phi^{(n)}_{n,\delta_F}(r)}{n!}\right|,$$

$$-\left|\ln r^{(n+1)\delta_F}\right| \le \ln\frac{\left|F^{(n)}(r)\right|}{n!} \le \left|\ln r^{(n+1)\delta_F}\right|,$$

$$-\left|(n+1)\delta_F \ln r\right| \le \ln\frac{\left|F^{(n)}(r)\right|}{n!} \le \left|(n+1)\delta_F \ln r\right|,$$

$$-(n+1)\delta_F|\ln r| \le \ln\frac{\left|F^{(n)}(r)\right|}{n!} \le (n+1)\delta_F|\ln r|,$$

$$e^{-(n+1)\delta_F|\ln r|} \le \frac{\left|F^{(n)}(r)\right|}{n!} \le e^{(n+1)\delta_F|\ln r|},$$

and thus inequalities (13) are proved.

Let us investigate equality signs in (13). Let for some $r = r_0 > 0$, $r_0 \neq 1$ and for some function $F(z) \in \widetilde{A}_n(\Pi)$ in the right part of (13) the equality sign occurs, i.e.

$$\frac{\left|F^{(n)}(r_0)\right|}{n!} = e^{(n+1)\delta_F|\ln r|} \tag{14}$$

or

$$\mathrm{Re}\left\{\nu_n\left[F(r_0)\right]\right\} = \ln\frac{\left|F^{(n)}(r_0)\right|}{n!} = \left|\ln r_0^{(n+1)\delta_F}\right| = \left|\ln\frac{\Phi^{(n)}_{n,\delta_F}(r_0)}{n!}\right|. \tag{15}$$

Using Theorem 10, we get

$$\mathrm{Re}\left\{\nu_n\left[F(r_0)\right]\right\} \le \left|\nu_n\left[F(r_0)\right]\right| \le \left|\nu_n\left[\Phi_{n,\delta_F}(r_0)\right]\right|.$$

Due to (15), come to equality (8):

$$\left|\nu_n\left[F(r_0)\right]\right| = \left|\nu_n\left[\Phi_{n,\delta_F}(r_0)\right]\right|.$$

Again using Theorem 10 we conclude, that the function $F^{(n)}(z)$ must be of the shape $F^{(n)}(z) = \Phi^{(n)}_{n,a}(z)$. After substituting it in (14) we will get

$$\frac{\left|\Phi^{(n)}_{n,a}(r_0)\right|}{n!} = e^{(n+1)|a||\ln r_0|},$$

or

$$\left|e^{(n+1)a\ln r_0}\right| = e^{(n+1)|a||\ln r_0|}.$$

Hence it follows that $a > 0$. Analogously is established equality sign on the left side of (13) □

Corollary 3. *If $F(z) \in \widetilde{U}_n(\Pi;\delta)$, then*

$$e^{-(n+1)\delta|\ln r|} \le \frac{\left|F^{(n)}(r)\right|}{n!} \le e^{(n+1)\delta|\ln r|}, \quad \forall r > 0.$$

Equality signs for $r = r_0 > 0$, $r_0 \neq 1$ realize only main functions $\Phi_{n,\pm\delta}(z) \in \widetilde{U}_n(\Pi;\delta)$.

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