

Expansions in Appell polynomials of the convolutions of probability distributions

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Abstract. We use the composition method to analyse the convolutions of probability distributions by employing the Appell polynomials and Bergström identity. Our approximation is based on the probability distributions which have the inverse generalized measure of bounded variation. The idea to use the accompanying probability distribution $e^{\lambda(F-E)}$, $\lambda > 0$, was first proposed by B.V. Gnedenko [1].

Keywords: convolution of probability distributions, accompanying law, Appell polynomials, Bergström identity.

1 Introduction

We consider the convolutions of one-dimensional distribution $F_{n1}, F_{n2}, \dots, F_{nk_n}$ defined by the formula

$$F_n = \prod_{j=1}^{k_n} F_{nj} = F_{n1} * \dots * F_{nk_n}$$

and analyse the asymptotic behavior of F_n , when $n \rightarrow \infty$ and $k_n \rightarrow \infty$. If the probability distribution

$$F = \frac{1}{k_n} \sum_{j=1}^{k_n} F_{nj} \tag{1}$$

has the inverse generalized measure F^- of bounded variation, i.e., $F * F^- = F^- * F = E$, where E is the degenerate at the point 0 probability distribution, then to evaluate the difference

$$F_n - F^{*k_n},$$

we use the results of Bergström [2]. In the case where F has no the above cited properties, we approximate F_n by infinitely divisible probability distribution

$$G_n = \exp \left\{ \sum_{j=1}^{k_n} (F_{nj} - E) \right\}.$$

Here is a short exposition of our work. Lemma 1 contains the estimate of the sum of the coefficients of Appell's polynomials. This new inequality is used in Theorem 3 to estimate the remainders of the expansion of the convolutions in Appell's polynomials. In (11), we present a new concept of the generalized Appell polynomials. In Theorem 4, we expand the convolution through the generalized Appell polynomials and find the convergence conditions. In Section 4, we present the expansion of the convolution $F_n = F_{n1} * \dots * F_{nk_n}$ in terms of the convolutions F^{*k_n} of identically distributed probability distributions F (see (1)). Theorem 7 contains the estimates of the remainders of such expansions. We intend to use this technique to analyze the convolutions of probability distributions defined on algebraic structures.

2 Some properties of the Appell polynomials

Let $g_n(x)$, $n = 0, 1, 2, \dots$, be a polynomial of order n . Denote by A the class

$$A = \left\{ g_n(x) : \frac{d}{dx} g_n(x) = g_{n-1}(x), \quad n = 1, 2, 3, \dots, \quad x \in \mathbb{R}^1 \right\},$$

named a polynomial Appell set (see [3] and [4, p. 242]).

It has been shown by Sheffer (see [4, 5]) that A is a polynomial set if and only if there exists a function of bounded variation $\beta(x)$, $x \in (0, +\infty)$, such that the integrals

$$b_n = \int_0^{\infty} x^n d\beta(x), \quad n = 0, 1, 2, \dots, \quad b_0 \neq 0,$$

converges and

$$g_n(x) = \int_0^{\infty} \frac{(x+t)^n}{n!} d\beta(t), \quad n = 0, 1, 2, \dots$$

Also, there exists the formal power series

$$B(t) = \sum_{n=0}^{\infty} a_n t^n, \quad a_0 \neq 0,$$

such that

$$B(t)e^{tx} = \sum_{n=0}^{\infty} g_n(x)t^n.$$

Often we will use Appell's polynomials $A_j(z)$, $j = 1, 2, \dots$, (see [3]) which are defined by the equality

$$e^{-z} \left(1 + \frac{z}{\tau}\right)^\tau = 1 + \sum_{j=1}^{\infty} \left(\frac{1}{\tau}\right)^j A_j(z), \quad |z| < \tau, \tau > 0,$$

here

$$A_j(z) = (-1)^j z^{j+1} \sum_{k=0}^{j-1} q_{j,k} z^k. \tag{2}$$

The coefficients $q_{j,k}$ satisfy the recurrent formula (see [6])

$$q_{j,k} = \frac{(j+k)q_{j-1,k} + q_{j-1,k-1}}{j+k+1}, \tag{3}$$

here $k = 1, 2, \dots, j-2$, $j = 1, 2, \dots$, and also

$$q_{j,0} = \frac{1}{j+1}, \quad q_{j,j-1} = \frac{1}{j!2^j},$$

where the coefficients $q_{j,k} = 0$, when $k < 0$.

We need the following lemma.

Lemma 1. *The following inequality is true:*

$$\sum_{k=0}^{j-1} q_{j,k} \leq \frac{1}{2}, \quad j = 1, 2, \dots$$

Proof. To proof the lemma, we will use the mathematical induction. Let

$$\sum_{k=0}^{j-1} q_{j,k} \leq \frac{1}{2}. \tag{4}$$

Now, by the use of recurrent formula (3), one obtains

$$\begin{aligned} \sum_{k=0}^j q_{j+1,k} &= q_{j+1,j} + q_{j+1,0} + \sum_{k=1}^{j-1} q_{j+1,k} \\ &= \frac{1}{2^{j+1}(j+1)!} + \frac{1}{j+2} + \sum_{k=1}^{j-1} \frac{(j+k+1)q_{j,k} + q_{j,k-1}}{j+k+2} \\ &= \frac{1}{2^{j+1}(j+1)!} + \frac{1}{j+2} - q_{j,0} \frac{j+1}{j+2} - q_{j,j-1} \frac{1}{2j+2} \\ &\quad + \sum_{k=0}^{j-1} q_{j,k} \left(\frac{j+k+1}{j+k+2} + \frac{1}{j+k+3} \right) \\ &= \sum_{k=0}^{j-1} q_{j,k} \left(1 - \frac{1}{j+k+2} + \frac{1}{j+k+3} \right) \leq \frac{1}{2}. \quad \square \end{aligned}$$

Note. J. Dickey (see [3]) has presented a table of some $q_{j,k}$ coefficients

j	k			
	1	2	3	4
1	1/2			
2	1/3	1/8		
3	1/4	1/6	1/48	
4	1/5	13/72	1/24	1/384

Remark 1. For all $j = 1, 2, \dots$ and $k = 0, 1, \dots, j - 1$,

$$0 \leq q_{j,k} \leq \frac{1}{2}.$$

Now we can estimate the remainder term

$$R_s(z, \tau) = e^{-z} \left(1 + \frac{z}{\tau}\right)^\tau - 1 - \sum_{j=1}^{s-1} \left(\frac{1}{\tau}\right)^j A_j(z), \quad s = 2, 3, \dots, |z| < \tau.$$

Lemma 2.

$$|R_s(z, \tau)| \leq \begin{cases} \frac{1}{2} \frac{|z|}{1-|z|/\tau} \left(\frac{|z|}{\tau}\right)^s & \text{for } |z| < \min(1, \tau), \\ \frac{1}{2} \frac{|z|}{|z|-1} \frac{1}{1-|z|^2/\tau} \left(\frac{|z|^2}{\tau}\right)^s & \text{for } 1 < |z| < \sqrt{\tau}. \end{cases}$$

Proof. Since

$$R_s(z, \tau) = \sum_{j=s}^{\infty} \left(-\frac{1}{\tau}\right)^j z^{j+1} \sum_{k=0}^{j-1} q_{j,k} z^k,$$

from Lemma 1 we have

$$|R_s(z, \tau)| \leq \sum_{j=s}^{\infty} \left(\frac{1}{\tau}\right)^j |z|^{j+1} \sum_{k=0}^{j-1} q_{j,k} \leq \frac{1}{2} \frac{|z|}{1-|z|/\tau} \left(\frac{|z|}{\tau}\right)^s,$$

when $|z| < \min(1, \tau)$.

In the case when $1 < |z| < \sqrt{\tau}$, we get

$$\begin{aligned} |R_s(z, \tau)| &\leq \sum_{j=s}^{\infty} \left(\frac{1}{\tau}\right)^j |z|^{j+1} \frac{1}{2} \sum_{k=0}^{j-1} |z|^k \leq \frac{|z|}{2} \sum_{j=s}^{\infty} \left(\frac{|z|}{\tau}\right)^j \frac{|z|^j}{|z|-1} \\ &\leq \frac{1}{2} \frac{|z|}{|z|-1} \frac{1}{1-|z|^2/\tau} \left(\frac{|z|^2}{\tau}\right)^s. \end{aligned} \quad \square$$

Remark 2. Assume that $|z| = 1$ and $\tau > 1$. Then

$$|R_s(z, \tau)| \leq \frac{1}{2} \frac{1}{\tau-1} \left(\frac{1}{\tau}\right)^{s-1}.$$

To analyse the expansions of probability distribution, Kalinin (see [7]) suggested to investigate the Newton binomial in form of $(1 + z/\sqrt[\nu]{\tau})^\tau$, when $\nu = 1, 2, \dots, \tau > 0$ and $|z| < \sqrt[\nu]{\tau}$.

By using the equality

$$\exp\left\{\sum_{j=1}^{\nu}(-1)^j \frac{z^j}{j} (\sqrt[\nu]{\tau})^{\nu-j}\right\} \left(1 + \frac{z}{\sqrt[\nu]{\tau}}\right)^\tau = 1 + \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt[\nu]{\tau}}\right)^j A_{j,\nu}(z)$$

for $|z| < \sqrt[\nu]{\tau}, \tau > 0, \nu = 1, 2, \dots$, we define the generalized Appell polynomial by

$$A_{j,\nu}(z) = (-1)^{j+\nu+1} z^{j+\nu} \sum_{k=0}^{j-1} (-1)^{(\nu+1)k} q_{j,k}^{(\nu)} z^{\nu k}. \tag{5}$$

Theorem 1. (See [7].) *The polynomials $A_{j,\nu}$ satisfy the recurrent formula*

$$A_{j,\nu}(y) = \int_0^y \left((-x)^\nu - x \frac{d}{dx} \right) A_{j-1,\nu}(x) dx,$$

where $A_{0,\nu}(y) = 1$ and $A_{j,\nu}$ have the form of (5). The coefficient $q_{j,k}^{(\nu)}$ satisfy the recurrent relation

$$q_{j,k}^{(\nu)} = \frac{(j + \nu(k + 1) - 1)q_{j-1,k}^{(\nu)} + q_{j-1,k-1}^{(\nu)}}{j + \nu(k + 1)},$$

where $k = 1, 2, \dots, j - 2$,

$$q_{j,j-1}^{(\nu)} = \frac{1}{j(\nu + 1)} q_{j-1,j-2}^{(\nu)} = \frac{1}{(\nu + 1)^j j!},$$

$$q_{j,0}^{(\nu)} = \frac{j + \nu - 1}{j + \nu} q_{j-1,0}^{(\nu)} = \frac{1}{\nu + j}.$$

Remark 3.

$$q_{j,k-1}^{(\nu)} = \sum' \frac{1}{\nu_1! \dots \nu_j!} \left(\frac{1}{\nu + 1}\right)^{\nu_1} \dots \left(\frac{1}{\nu + j}\right)^{\nu_j}, \quad k = 1, 2, \dots, j,$$

where the symbol \sum' denotes summation over all non-negative integer solutions of the system of equations

$$\nu_1 + 2\nu_2 + \dots + j\nu_j = j,$$

$$\nu_1 + \nu_2 + \dots + \nu_j = k.$$

Remark 4.

$$A_{j,1}(z) = A_j(z) \quad \text{and} \quad q_{j,k-1}^{(1)} = q_{j,k-1}.$$

Applying (2) and (3) once again, we obtain the following assertion.

Lemma 3.

$$\sum_{k=0}^{j-1} q_{j,k}^{(\nu)} \leq \frac{1}{\nu+1}, \quad j = 1, 2, \dots, \nu = 1, 2, \dots$$

Now we shall consider

$$R_s^{(\nu)}(z, \tau) = \exp \left\{ \sum_{j=1}^{\nu} (-1)^j \frac{z^j}{j} (\sqrt[\nu]{\tau})^{\nu-j} \right\} \left(1 + \frac{z}{\sqrt[\nu]{\tau}} \right)^{\tau} - 1 - \sum_{j=1}^{s-1} \left(\frac{1}{\sqrt[\nu]{\tau}} \right)^j A_{j,\nu}(z),$$

where $s = 2, 3, \dots$

Lemma 4.

$$|R_s^{(\nu)}(z, \tau)| \leq \begin{cases} \frac{|z|^\nu}{(\nu+1)(1-|z|/\sqrt[\nu]{\tau})} \left(\frac{|z|}{\sqrt[\nu]{\tau}} \right)^s & \text{for } |z| < \min(1, \sqrt[\nu]{\tau}), \\ \frac{|z|^\nu}{(\nu+1)(|z|^\nu-1)(1-|z|^{\nu+1}/\sqrt[\nu]{\tau})} \left(\frac{|z|^{\nu+1}}{\sqrt[\nu]{\tau}} \right)^s & \text{for } 1 < |z| < \tau^{1/(\nu(\nu+1))}, \end{cases}$$

where $\nu = 1, 2, \dots$ and $s = 2, 3, \dots$

Proof. Since

$$R_s^{(\nu)}(z, \tau) = \sum_{j=s}^{\infty} \left(\frac{1}{\sqrt[\nu]{\tau}} \right)^j A_{j,\nu}(z) = \sum_{j=s}^{\infty} \left(-\frac{1}{\sqrt[\nu]{\tau}} \right)^j z^{j+\nu} \sum_{k=0}^{j-1} (-1)^{(\nu+1)(k+1)} q_{j,k}^{(\nu)} z^{\nu k},$$

from Lemma 3 we have

$$|R_s^{(\nu)}(z, \tau)| \leq \sum_{j=s}^{\infty} \left(\frac{1}{\sqrt[\nu]{\tau}} \right)^j |z|^{j+\nu} \sum_{k=0}^{j-1} q_{j,k}^{(\nu)} \leq \frac{|z|^\nu}{\nu+1} \frac{1}{1-|z|/\sqrt[\nu]{\tau}} \left(\frac{|z|}{\sqrt[\nu]{\tau}} \right)^s,$$

when $|z| < \min(1, \sqrt[\nu]{\tau})$.

In the case when $1 < |z| < \tau^{1/(\nu(\nu+1))}$, we get

$$\begin{aligned} |R_s^{(\nu)}(z, \tau)| &\leq \sum_{j=s}^{\infty} \left(\frac{1}{\sqrt[\nu]{\tau}} \right)^j |z|^{j+\nu} \frac{1}{\nu+1} \sum_{k=0}^{j-1} |z|^{\nu k} \leq \frac{|z|^\nu}{\nu+1} \sum_{j=s}^{\infty} \left(\frac{|z|}{\sqrt[\nu]{\tau}} \right)^j \frac{|z|^{\nu j}}{|z|^\nu-1} \\ &= \frac{|z|^\nu}{(\nu+1)(|z|^\nu-1)(1-|z|^{\nu+1}/\sqrt[\nu]{\tau})} \left(\frac{|z|^{\nu+1}}{\sqrt[\nu]{\tau}} \right)^s. \quad \square \end{aligned}$$

Remark 5. For $|z| = 1$ and $\tau > 1$, one has

$$|R_s^{(\nu)}(z, \tau)| \leq \frac{1}{\nu+1} \frac{1}{\sqrt[\nu]{\tau}-1} \left(\frac{1}{\sqrt[\nu]{\tau}} \right)^{s-1}.$$

3 The expansion of convolution F^{*n} in Appell polynomials

In our work, we replace the exponent, for example $(1 + z/n)^n$, with the convolution $(E + (1/n)\mu)^{*n}$, where $n = 1, 2, \dots$, and μ is the generalized measure of bounded variation, i.e.,

$$\left(E + \frac{\mu}{n}\right)^{*n} = e^\mu * \left(E + \sum_{j=1}^{\infty} \left(\frac{1}{n}\right)^j A_j(\mu)\right),$$

here $A_j(\mu)$ is the Appell polynomial (see [8])

$$A_j(\mu) = (-1)^j \mu^{*(j+1)} * \sum_{k=0}^{j-1} q_{j,k} \mu^{*k}.$$

In what follows, $|z|$ is replaced with the bounded variation of μ (we also use $\mathbf{V}(g) = \mathbf{V}(g(x))$ to denote the full variation of the function $g(x)$), i.e.,

$$|z|^j \rightarrow (\mathbf{V}(\mu))^j, \quad j = 0, 1, 2, \dots$$

Let the probability distribution Q has the inverse measure of bounded variation Q^- , i.e., $Q * Q^- = Q^- * Q = E$.

Theorem 2. (See [9].) *Let*

$$\mathbf{V}(\sqrt{n}(F - Q) * Q^-) < 1.$$

Then the following formal expansion holds true:

$$F^{*n} = Q^{*n} * e^{n(F-Q)*Q^-} * \left(E + \sum_{j=1}^{\infty} \left(\frac{1}{n}\right)^j A_j(n(F - Q) * Q^-)\right).$$

Here

$$\begin{aligned} &A_j(n(F - Q) * Q^-) \\ &= (-1)^j (n(F - Q) * Q^-)^{*(j+1)} * \sum_{k=0}^{j-1} q_{j,k} (n(F - Q) * Q^-)^{*k} \end{aligned} \tag{6}$$

are the Appell polynomials and

$$q_{j,k} = \sum' \frac{1}{\nu_1! \dots \nu_j!} \left(\frac{1}{2}\right)^{\nu_1} \dots \left(\frac{1}{j+1}\right)^{\nu_j}, \quad k = 0, 1, 2, \dots, j - 1,$$

where \sum' denotes summation over all solutions in non-negative integers ν_1, \dots, ν_j of the following system of two Diophantine equations

$$\begin{aligned} \nu_1 + 2\nu_2 + \dots + j\nu_j &= j, \\ \nu_1 + \nu_2 + \dots + \nu_j &= k + 1. \end{aligned}$$

The proof of Theorem 2 is based on the estimate of the remainder term

$$r_n^{(s)}(B-x) = \sum_{j=s}^{\infty} A_j (n(F-Q) * Q^-) * Q^{*n} * e^{n(F-Q)*Q^-}(B-x), \quad (7)$$

for any Borel set $B \subset \mathbb{R}^1$ given in

Theorem 3. *Let*

$$\rho = \mathbf{V}((F-Q) * Q^-)$$

and $n\rho^2 < 1$, $n > 1$. Then, for any Borel set B , the following inequality holds:

$$\sup_x |r_n^{(s)}(B-x)| \leq \sup_x |n(F-Q)^{*(s+1)} * Q^{*(n-s-1)} * e^{n(F-Q)*Q^-}(B-x)| \Delta_n,$$

where

$$\Delta_n \leq \begin{cases} \frac{1}{2(1-\rho)} \frac{1}{1-n\rho}, & \text{when } n\rho < 1, \\ \frac{1}{2(1-\rho)}, & \text{when } n\rho = 1, \\ \frac{(n\rho)^s}{2(n\rho-1)} \frac{1}{1-n\rho^2}, & \text{when } \frac{1}{n} < \rho < \frac{1}{\sqrt{n}}. \end{cases}$$

Proof. From (6) and (7) it follows that

$$\begin{aligned} & \sup_x |r_n^{(s)}(B-x)| \\ &= \sup_x \left| Q^{*n} * e^{n(F-Q)*Q^-} * \sum_{j=s}^{\infty} (-1)^j \left(\frac{1}{n}\right)^j (n(F-Q) * Q^-)^{*(j+1)} \right. \\ & \quad \left. * \sum_{k=0}^{j-1} q_{j,k}(n(F-Q) * Q^-)^{*k}(B-x) \right| \\ &= \sup_x \left| n(F-Q)^{*(s+1)} * Q^{*(n-s-1)} * e^{n(F-Q)*Q^-} \right. \\ & \quad \left. * \sum_{j=s}^{\infty} (-1)^j ((F-Q) * Q^-)^{*(j-s)} * \sum_{k=0}^{j-1} q_{j,k}(n(F-Q) * Q^-)^{*k}(B-x) \right| \\ &\leq \sup_x |n(F-Q)^{*(s+1)} * Q^{*(n-s-1)} * e^{n(F-Q)*Q^-}(B-x)| \\ & \quad \times \sum_{j=s}^{\infty} (\mathbf{V}((F-Q) * Q^-))^{j-s} \sum_{k=0}^{j-1} q_{j,k} (\mathbf{V}(n(F-Q) * Q^-))^k. \end{aligned} \quad (8)$$

To estimate the sum

$$\Delta_n = \sum_{j=s}^{\infty} \rho^{j-s} \sum_{k=0}^{j-1} q_{j,k} (n\rho)^k, \quad (9)$$

we use the inequality (see (4))

$$\sum_{k=1}^{j-1} q_{j,k} \leq \frac{1}{2},$$

for all $j = 1, 2, \dots$. It is evident that

$$\sum_{k=0}^{j-1} q_{j,k} (n\rho)^k \leq \begin{cases} \frac{1}{2} & \text{if } n\rho \leq 1, \\ \frac{1}{2} \frac{(n\rho)^j}{n\rho-1} & \text{otherwise.} \end{cases}$$

Thus, for $n > 1$, we have

$$\Delta_n \leq \begin{cases} \frac{1}{2(1-\rho)} & \text{if } n\rho \leq 1, \\ \frac{(n\rho)^s}{2(n\rho-1)} \frac{1}{1-n\rho^2} & \text{if } \frac{1}{n} < \rho < \frac{1}{\sqrt{n}}. \end{cases} \tag{10}$$

Now the assertion of Theorem 2 follows immediately from inequalities (8)–(10). \square

The generalized Appell polynomials, whose argument are the convolutions of measure μ are defined as

$$A_{j,s}(\mu) = (-1)^{j+s+1} \mu^{*(j+s)} * \sum_{k=0}^{j-1} (-1)^{(s+1)k} q_{j,k}^{(s)} \mu^{*(sk)}, \tag{11}$$

where

$$q_{j,k}^{(s)} = \sum' \frac{1}{\nu_1! \dots \nu_j!} \left(\frac{1}{s+1}\right)^{\nu_1} \dots \left(\frac{1}{s+j}\right)^{\nu_j}.$$

The exposition of the properties of polynomials $A_{j,s}(\mu)$ is given by V.M. Kalinin [7, Thm. 2.1 and Corols. 2.3, 2.4].

Note that if $\mathbf{V}(\mu) < \sqrt[s]{n}$ and μ is a generalized measure of bounded variation then the formal equality

$$\left(E + \frac{\mu}{\sqrt[s]{n}}\right)^{*n} = \exp\left\{\sum_{j=1}^s \frac{(-1)^{j+1}}{j} (\sqrt[s]{n})^{s-j} \mu^{*j}\right\} * \left(E + \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt[s]{n}}\right)^j A_{j,s}(\mu)\right)$$

holds. This claim can be proved as follows:

$$\begin{aligned} & \left(E + \frac{\mu}{\sqrt[s]{n}}\right)^{*n} \\ &= \exp\left\{n \log\left(E + \frac{\mu}{\sqrt[s]{n}}\right)\right\} = \exp\left\{\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (\sqrt[s]{n})^{s-j} \mu^{*j}\right\} \\ &= \exp\left\{\sum_{j=1}^s \frac{(-1)^{j+1}}{j} (\sqrt[s]{n})^{s-j} \mu^{*j}\right\} * \exp\left\{\sum_{j=1}^{\infty} \frac{(-1)^{j+s+1}}{j+s} \left(\frac{1}{\sqrt[s]{n}}\right)^j \mu^{*(j+s)}\right\}. \end{aligned}$$

Here

$$\exp\left\{\sum_{j=1}^{\infty} \frac{(-1)^{j+s+1}}{j+s} \left(\frac{1}{\sqrt[s]{n}}\right)^j \mu^{*(j+s)}\right\} = E + \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt[s]{n}}\right)^j A_{j,s}(\mu).$$

We apply (11) to probability measures Q , which have the inverse measures Q^- and get

$$\begin{aligned} F^{*n} &= Q^{*n} * \left(E + \frac{\sqrt[s]{n}(F-Q) * Q^-}{\sqrt[s]{n}}\right)^{*n} \\ &= Q^{*n} * \exp\left\{\sum_{j=1}^s \frac{(-1)^j}{j} (\sqrt[s]{n})^{s-j} (\sqrt[s]{n}(F-Q) * Q^-)^{*j}\right\} \\ &\quad * \left(E + \sum_{j=1}^{\infty} \left(\frac{1}{\sqrt[s]{n}}\right)^j A_{j,s}(\sqrt[s]{n}(F-Q) * Q^-)\right). \end{aligned} \quad (12)$$

In this asymptotic equality, we shall estimate the remainder term

$$\bar{r}_n^{(s+1)} = \sum_{j=s}^{\infty} \left(\frac{1}{\sqrt[s]{n}}\right)^j A_{j,s}(\sqrt[s]{n}(F-Q) * Q^-) * G_n^{(s)}. \quad (13)$$

Here

$$G_n^{(s)} = Q^{*n} * \exp\left\{\sum_{j=1}^s \frac{(-1)^j}{j} (\sqrt[s]{n})^{s-j} (\sqrt[s]{n}(F-Q) * Q^-)^{*j}\right\}.$$

Theorem 4. *Let*

$$\rho = \mathbf{V}((F-Q) * Q^-)$$

and $n\rho^{s+1} < 1$, when $n > s$. Then, for any Borel set B the following inequality holds:

$$\sup_x |\bar{r}_n^{(s+1)}(B-x)| \leq \sup_x |nG_n^{(s)} * ((F-Q) * Q^-)^{*(2s)}(B-x)| \Delta_n^{(s)},$$

where

$$\Delta_n^{(s)} = \sum_{j=1}^{\infty} \rho^{j-1} \sum_{k=0}^{j-1} q_{j,k}^{(s)} (n\rho^s)^k$$

and

$$\Delta_n^{(s)} \leq \begin{cases} \frac{1}{s+1} \frac{1}{1-\rho} & \text{if } n\rho^s \leq 1 \\ \frac{n\rho^s}{(s+1)(n\rho^s-1)} \frac{1}{1-n\rho^{s+1}} & \text{if } \left(\frac{1}{n}\right)^{1/s} < \rho < \left(\frac{1}{n}\right)^{1/(s+1)}. \end{cases}$$

Proof. From (11), (12) and (13) it follows that

$$\begin{aligned} \bar{r}_n^{(s+1)} &= \sum_{j=s}^{\infty} (-1)^{j+s+1} \left(\frac{1}{\sqrt[s]{n}}\right)^j (\sqrt[s]{n}(F - Q) * Q^-)^{*(j+s)} \\ &\quad * \sum_{k=0}^{j-1} (-1)^{(s+1)k} q_{j,k}^{(s)} (\sqrt[s]{n}(F - Q) * Q^-)^{*(sk)} * G_n^{(s)} \\ &= \left(\frac{1}{\sqrt[s]{n}}\right)^s (\sqrt[s]{n}(F - Q) * Q^-)^{*(s)} * G_n^{(s)} \\ &\quad * \sum_{j=s}^{\infty} (-1)^{j+s+1} \left(\frac{1}{\sqrt[s]{n}}\right)^{j-s} (\sqrt[s]{n}(F - Q) * Q^-)^{*(j-s)} \\ &\quad * \sum_{k=0}^{j-1} (-1)^{(s+1)k} q_{j,k}^{(s)} (\sqrt[s]{n}(F - Q) * Q^-)^{*(sk)}. \end{aligned} \tag{14}$$

For all Borel sets B , we get

$$\sup_x |\bar{r}_n^{(s+1)}(B - x)| \leq \sup_x |n((F - Q) * Q^-)^{*(2s)} * G_n^{(s)}(B - x)| \Delta_n^{(s)}. \tag{15}$$

By using Lemma 3, we get

$$\begin{aligned} \Delta_n^{(s)} &= \sum_{j=s}^{\infty} \rho^{j-s} \sum_{k=0}^{j-1} q_{j,k}^{(s)} (n\rho^s)^k \\ &\leq \begin{cases} \frac{1}{s+1} \frac{1}{1-\rho} & \text{if } \rho \leq \left(\frac{1}{n}\right)^{1/s}, \\ \frac{n\rho^s}{(s+1)(n\rho^s-1)} \frac{1}{1-n\rho^{s+1}} & \text{if } \left(\frac{1}{n}\right)^{1/s} < \rho < \left(\frac{1}{n}\right)^{1/(s+1)}. \end{cases} \end{aligned} \tag{16}$$

Theorem 4 follows from (14)–(16). □

4 The expansion of convolutions $F_{n_1} * \dots * F_{n_{k_n}}$

The asymptotic expansions of the Bergström type convolutions $F_n = F_{n_1} * \dots * F_{n_{k_n}}$ were analyzed in [10–13]. We use the identity (see [9])

$$F_{n_j} = e^{F_{n_j} - E} * (E - (F_{n_j} - E))^{*2} * \mathbf{E}(E - F_{n_j})^{*\mu_1}$$

and get

$$F_n = \exp\left\{\sum_{j=1}^{k_n} (F_{n_j} - E)\right\} * \prod_{j=1}^{k_n} (E - (F_{n_j} - E))^{*2} * \mathbf{E}(E - F_{n_j})^{*\mu_1}, \tag{17}$$

where $E = E(x)$ is the degenerate at point $x = 0$ probability distribution,

$$\mathbf{E}(E - F_{nj})^{*\mu_1} = \sum_{k=0}^{\infty} \mathbf{P}\{\mu_1 = k\} (E - F_{nj})^{*k},$$

and

$$\mathbf{P}\{\mu_1 = k\} = \frac{k+1}{(k+2)!}, \quad k = 0, 1, \dots$$

Note that, in (17)

$$Q_j = -(F_{nj} - E)^{*2} * \mathbf{E}(E - F_{nj})^{*\mu_1}, \quad j = 0, 1, \dots, k_n, \quad (18)$$

is generalized measure of bounded variation.

Set

$$\begin{aligned} M_1(Q) &= \sum_{j=1}^{k_n} Q_j \\ M_2(Q) &= \sum_{1 \leq j_1 < j_2 \leq k_n} Q_{j_1} * Q_{j_2} \\ &\vdots \\ M_m(Q) &= \sum_{1 \leq j_1 < j_2 < \dots < j_m \leq k_n} Q_{j_1} * Q_{j_2} * \dots * Q_{j_m} \\ &\vdots \\ M_{k_n}(Q) &= \prod_{j=1}^{k_n} Q_j. \end{aligned}$$

Lemma 5. *The following identity is true:*

$$\prod_{j=1}^{k_n} (E + Q_j) = E + \sum_{m=1}^{k_n} M_m(Q). \quad (19)$$

Proof. To prove this lemma, we shall use the mathematical induction.

If $k_n = 1$, then $M_1(Q) = Q_1$.

Set

$$Q'_j = Q_1 * \dots * Q_{j-1} * Q_{j+1} * \dots * Q_{s+1}, \quad 1 \leq j \leq k_n - 1.$$

Suppose that identity (19) is true with $k_n = s$, i.e.,

$$\prod_{j=1}^s (E + Q_j) = E + \sum_{m=1}^s M_m(Q'_{s+1}).$$

We want to prove that it is also true, when $k_n = s + 1$, i.e.,

$$\prod_{j=1}^{s+1} (E + Q_j) = E + \sum_{m=1}^{s+1} M_m(Q).$$

We have

$$\begin{aligned}
 \prod_{j=1}^{s+1} (E + Q_j) &= \prod_{j=1}^s (E + Q_j) * (E + Q_{s+1}) \\
 &= \left(E + \sum_{m=1}^s M_m(Q'_{s+1}) \right) * (E + Q_{s+1}) \\
 &= E + \sum_{m=1}^s M_m(Q'_{s+1}) + Q_{s+1} + \sum_{m=1}^s Q_{s+1} * M_m(Q'_{s+1}) \\
 &= E + (M_1(Q'_{s+1}) + Q_{s+1}) \\
 &\quad + \sum_{m=2}^s (M_m(Q'_{s+1}) + Q_{s+1} * M_{m-1}(Q'_{s+1})) + Q_{s+1} * M_s(Q'_{s+1}) \\
 &= E + M_1(Q) + \sum_{m=2}^s M_m(Q) + M_{s+1}(Q) = E + \sum_{m=1}^{s+1} M_m(Q).
 \end{aligned}$$

Here

$$\begin{aligned}
 M_1(Q'_{s+1}) + Q_{s+1} &= \sum_{j=1}^s Q_j + Q_{s+1} = M_1(Q), \\
 M_m(Q'_{s+1}) + Q_{s+1} * M_{m-1}(Q'_{s+1}) \\
 &= \sum_{1 \leq j_1 < j_2 < \dots < j_m \leq s} Q_{j_1} * Q_{j_2} * \dots * Q_{j_m} \\
 &\quad + \sum_{1 \leq j_1 < j_2 < \dots < j_{m-1} \leq s} Q_{j_1} * Q_{j_2} * \dots * Q_{j_{m-1}} * Q_{s+1} \\
 &= \sum_{1 \leq j_1 < j_2 < \dots < j_m \leq s+1} Q_{j_1} * Q_{j_2} * \dots * Q_{j_m} = M_m(Q), \quad 2 \leq m \leq s, \\
 Q_{s+1} * M_s(Q'_{s+1}) &= Q_{s+1} * \prod_{j=1}^s Q_j = \prod_{j=1}^{s+1} Q_j = M_{s+1}(Q). \quad \square
 \end{aligned}$$

Theorem 5. *It is true formal identity*

$$\begin{aligned}
 F_n &= e^{G-E} * \left(E + \sum_{m=1}^s M_m(Q) \right) + R_n^{(s+1)}, \\
 R_n^{(s+1)} &= e^{G-E} * \sum_{m=s+1}^n M_m(Q),
 \end{aligned}$$

where $G = \sum_{j=1}^{k_n} F_{nj}$.

Proof. The theorem follows from (17)–(19). □

Now we shall approximate F_n by using the convolutions of probability distributions $G_n = G_{n1} * \dots * G_{nk_n}$ such that all G_{nj} posses the inverse generalized measures G_{nj}^- , $j = 1, 2, \dots, k_n$, i.e., $G_{nj} * G_{nj}^- = G_{nj}^- * G_{nj} = E$ of bounded variation. We shell also employ the generalized Appell polynomials.

Let $\rho = \sup_{1 \leq j \leq k_n} \mathbf{V}((F_{nj} - G_{nj}) * G_{nj}^-)$ and $k_n \rho^s < 1$. The following formal expansion holds:

$$\begin{aligned} F_n &= G_n * \prod_{j=1}^{k_n} (E + (F_{nj} - G_{nj}) * G_{nj}^-) \\ &= G_n * \exp \left\{ \sum_{j=1}^{k_n} \log (E + (F_{nj} - G_{nj}) * G_{nj}^-) \right\} \\ &= G_n * \exp \left\{ \sum_{m=1}^s \frac{(-1)^{m+1}}{m} \sum_{j=1}^{k_n} ((F_{nj} - G_{nj}) * G_{nj}^-)^{*m} \right\} \\ &\quad * \exp \left\{ \sum_{l=1}^{\infty} \frac{(-1)^{l+s+1}}{l+s} \left(\frac{1}{\sqrt[s]{k_n}} \right)^l (\sqrt[s]{k_n})^l \sum_{j=1}^{k_n} ((F_{nj} - G_{nj}) * G_{nj}^-)^{*(l+s)} \right\}. \end{aligned}$$

Define

$$\alpha_l = \frac{(-1)^{l+s+1}}{l+s} (\sqrt[s]{k_n})^l \sum_{j=1}^{k_n} ((F_{nj} - G_{nj}) * G_{nj}^-)^{*(l+s)}, \tag{20}$$

$$D_n^{(s)} = G_n * \exp \left\{ \sum_{m=1}^s (\sqrt[s]{k_n})^{s-m} \alpha_{m-s} \right\},$$

$$R_n^{(s+1)} = \sum_{r=1}^{\infty} \left(\frac{1}{\sqrt[s]{k_n}} \right)^r \beta_r * D_n^{(s)}, \tag{21}$$

where

$$\beta_r = \sum_{\substack{\nu_1+2\nu_2+\dots+r\nu_r=r \\ \nu_1+\nu_2+\dots+\nu_r=k \\ k=1,2,\dots,r}} \frac{1}{\nu_1! \dots \nu_r!} \alpha_1^{*\nu_1} * \dots * \alpha_r^{*\nu_r}. \tag{22}$$

Theorem 6. Let $k_n \rho^s < 1$. Then, for all Borel sets $B \subset \mathbb{R}^1$,

$$F_n(B) = G_n * \exp \left\{ \sum_{m=1}^s \frac{(-1)^{m+1}}{m} \sum_{j=1}^{k_n} ((F_{nj} - G_{nj}) * G_{nj}^-)^{*m} \right\} (B) + R_n^{(s+1)}(B),$$

where

$$\sup_x |R_n^{(s+1)}(B - x)| \leq \Delta_n^{(s)}(B) \frac{\rho^s k_n}{(s+1)(1-\rho)}$$

for $\sqrt[s]{k_n}\rho \leq 1$ and

$$\sup_x |R_n^{(s+1)}(B-x)| \leq \Delta_n^{(s)}(B) \frac{(\sqrt[s]{k_n}\rho)^{s+1}}{(\sqrt[s]{k_n}\rho - 1)(1 - \sqrt[s]{k_n}\rho^2)}$$

for $(1/\sqrt[s]{k_n})^2 < \rho^2 < 1/\sqrt[s]{k_n}$. Here

$$\Delta_n^{(s)}(B) = \sum_{j=1}^{k_n} \sup_x |((F_{nj} - G_{nj}) * G_{nj}^-)^{(s+1)} * D_n^{(s)}(B-x)|.$$

Proof. We have

$$J = \exp \left\{ \sum_{l=1}^{\infty} \left(\frac{1}{\sqrt[s]{k_n}} \right)^l \alpha_l \right\} - E = \sum_{r=1}^{\infty} \left(\frac{1}{\sqrt[s]{k_n}} \right)^r \beta_r.$$

We get

$$\mathbf{V}(J) \leq \sum_{r=1}^{\infty} \left(\frac{1}{\sqrt[s]{k_n}} \right)^r \sum_{\substack{\nu_1+2\nu_2+\dots+r\nu_r=r \\ \nu_1+\nu_2+\dots+\nu_r=k \\ k=1,2,\dots,r}} \frac{1}{\nu_1! \dots \nu_r!} \prod_{l=1}^r (\mathbf{V}(\alpha_l))^{\nu_l}, \tag{23}$$

where

$$\begin{aligned} \mathbf{V}(\alpha_l) &\leq \frac{1}{l+s} (\sqrt[s]{k_n})^l \sum_{j=1}^{k_n} (\mathbf{V}((F_{nj} - G_{nj}) * G_{nj}^-))^{l+s} \\ &\leq \frac{1}{l+s} (\sqrt[s]{k_n})^l k_n \rho^{l+s} = \frac{(\sqrt[s]{k_n}\rho)^{l+s}}{l+s}. \end{aligned} \tag{24}$$

From (23) and (24) it follows that

$$\begin{aligned} \mathbf{V}(J) &\leq \sum_{r=1}^{\infty} \left(\frac{1}{\sqrt[s]{k_n}} \right)^r \\ &\times \sum_{\substack{\nu_1+2\nu_2+\dots+r\nu_r=r \\ \nu_1+\nu_2+\dots+\nu_r=k \\ k=1,2,\dots,r}} \frac{1}{\nu_1! \dots \nu_r!} \left(\frac{1}{s+1} \right)^{\nu_1} \dots \left(\frac{1}{s+r} \right)^{\nu_r} (\sqrt[s]{k_n}\rho)^{\sum_{l=1}^r \nu_l(l+s)}. \end{aligned}$$

From Lemma 3 it follows that

$$\sum_{k=1}^r \sum_{\substack{\nu_1+\dots+\nu_r=k \\ \nu_1+2\nu_2+\dots+r\nu_r=r}} \frac{1}{\nu_1! \dots \nu_r!} \prod_{l=1}^r \left(\frac{1}{s+l} \right)^{\nu_l} \leq \frac{1}{s+1}.$$

Note that in the case when $\sqrt[s]{k_n}\rho \leq 1$ and $k_n > 1$, we have

$$\mathbf{V}(J) \leq \sum_{r=1}^{\infty} \rho^r \frac{(\sqrt[s]{k_n})^s}{s+1} \leq \frac{k_n \rho^{s+1}}{(s+1)(1-\rho)}. \tag{25}$$

Similarly, in the case when $\sqrt[s]{k_n\rho} > 1$ and $\sqrt[s]{k_n\rho^2} < 1$,

$$\begin{aligned} \mathbf{V}(J) &\leq \sum_{r=1}^{\infty} \rho^r (\sqrt[s]{k_n\rho})^s \frac{(\sqrt[s]{k_n\rho})^r}{\sqrt[s]{k_n\rho} - 1} \\ &= \frac{(\sqrt[s]{k_n\rho})^s}{\sqrt[s]{k_n\rho} - 1} \sum_{r=1}^{\infty} (\sqrt[s]{k_n\rho^2})^r = \frac{(\sqrt[s]{k_n\rho})^s}{\sqrt[s]{k_n\rho} - 1} \frac{\sqrt[s]{k_n\rho^2}}{1 - \sqrt[s]{k_n\rho^2}}. \end{aligned} \quad (26)$$

From (20)–(22) it follows that

$$R_n^{(s+1)} = \sum_{r=1}^{\infty} \left(\frac{1}{\sqrt[s]{k_n}} \right)^r \sum_{\substack{\nu_1+2\nu_2+\dots+r\nu_r=r \\ \nu_1+\nu_2+\dots+\nu_r=k \\ k=1,2,\dots,r}} \prod_{\substack{l=1 \\ l \neq l_0}}^r \alpha_l^{*\nu_l} * \alpha_{l_0}^{*\nu_{l_0}} * D_n^{(s)},$$

where $\nu_{l_0} > 1$. Here

$$\begin{aligned} \alpha_{l_0}^{*\nu_{l_0}} * D_n^{(s)} &= \frac{(-1)^{l_0+s+1}}{l_0+s} (\sqrt[s]{k_n})^{l_0} \sum_{j=1}^{k_n} ((F_{nj} - G_{nj}) * G_{nj}^-)^{*(s+1)} \\ &\quad * D_n^{(s)} * ((F_{nj} - G_{nj}) * G_{nj}^-)^{*(l_0-1)} \end{aligned}$$

and for all Borel sets $B \subset \mathbb{R}^1$

$$\begin{aligned} &\alpha_{l_0}^{*\nu_{l_0}} * D_n^{(s)}(B) \\ &= \frac{(-1)^{l_0+s+1}}{l_0+s} (\sqrt[s]{k_n})^{l_0} \sum_{j=1}^{k_n} \int_{-\infty}^{\infty} ((F_{nj} - G_{nj}) * G_{nj}^-)^{*(s+1)} \\ &\quad * D_n^{(s)}(B-x) ((F_{nj} - G_{nj}) * G_{nj}^-)^{*(l_0-1)}(dx). \end{aligned}$$

Thus

$$\begin{aligned} &\sup_x |\alpha_{l_0}^{*\nu_{l_0}} * D_n^{(s)}(B-x)| \\ &\leq \frac{(\sqrt[s]{k_n})^{l_0}}{l_0+s} \sum_{j=1}^{k_n} \rho^{l_0-1} \sup_x |((F_{nj} - G_{nj}) * G_{nj}^-)^{*(s+1)} * D_n^{(s)}(B-x)| \\ &= \frac{(\sqrt[s]{k_n\rho})^{l_0}}{l_0+s} \frac{1}{\rho} \sum_{j=1}^{k_n} \sup_x |((F_{nj} - G_{nj}) * G_{nj}^-)^{*(s+1)} * D_n^{(s)}(B-x)| \end{aligned}$$

and

$$\begin{aligned} &\sup_x |R_n^{(s+1)}(B-x)| \\ &\leq \frac{1}{\rho} \sum_{j=1}^{k_n} \sup_x |((F_{nj} - G_{nj}) * G_{nj}^-)^{*(s+1)} * D_n^{(s)}(B-x)| \mathbf{V}(J). \end{aligned}$$

From the above and (25), (26) it follows that

$$\begin{aligned} & \sup_x |R_n^{(s+1)}(B-x)| \\ &= \begin{cases} \Delta_n^{(s)}(B) \frac{k_n \rho^s}{(s+1)(1-\rho)} & \text{if } \sqrt[s]{k_n} \rho \leq 1, \\ \Delta_n^{(s)}(B) \frac{(\sqrt[s]{k_n} \rho)^{s+1}}{(\sqrt[s]{k_n} \rho - 1)(1 - \sqrt[s]{k_n} \rho^2)} & \text{if } \left(\frac{1}{\sqrt[s]{k_n}}\right)^2 < \rho^2 < \frac{1}{\sqrt[s]{k_n}}. \end{cases} \quad \square \end{aligned}$$

Let

$$G = \frac{1}{k_n} \sum_{j=1}^{k_n} F_{nj} \tag{27}$$

and

$$F_n - G^{k_n} = \prod_{j=1}^{k_n} F_{nj} - \left(\frac{1}{k_n} \sum_{j=1}^{k_n} F_{nj} \right)^{*k_n}.$$

Suppose that G has the generalized inverse measure G^- of bounded variation. The following claim is a corollary of Theorem 6 with $s = 1$ and $G = G_{n1} = \dots = G_{nk_n}$.

Theorem 7. *Let*

$$\rho = \sup_{1 \leq j \leq k_n} \mathbf{V}((F_{nj} - G) * G^-),$$

where $k_n \rho < 1$ and $k_n > 1$, then, for all Borel sets $B \subset \mathbb{R}^1$,

$$\prod_{j=1}^{k_n} F_{nj}(B) = \left(\frac{1}{k_n} \sum_{j=1}^{k_n} F_{nj} \right)^{*k_n}(B) + R_n^{(2)}(B),$$

where

$$\sup_x |R_n^{(2)}(B-x)| \leq \frac{k_n \rho}{2(1-\rho)} \sum_{j=1}^{k_n} \sup_x |((F_{nj} - G) * G^-)^{*2} * G^{*k_n}(B-x)|$$

if $k_n \rho < 1$, and

$$\begin{aligned} & \sup_x |R_n^{(2)}(B-x)| \\ & \leq \frac{k_n \rho^2}{(k_n \rho - 1)(1 - k_n \rho^2)} \sum_{j=1}^{k_n} \sup_x |((F_{nj} - G) * G^-)^{*2} * G^{*k_n}(B-x)| \end{aligned}$$

if $1/k_n < \rho < 1/\sqrt{k_n}$.

Proof. From the definition of probability distribution G in (27), it follows that

$$\sum_{j=1}^{k_n} (F_{nj} - G) \equiv 0 \quad \text{and} \quad G_n = G^{*k_n}.$$

Thus, we have

$$D_n^{(1)} = G^{*k_n} \quad \text{and} \quad \Delta_n^{(1)}(B) = \sum_{j=1}^{k_n} \sup_x |((F_{n_j} - G) * G^-)^{*2} * G^{*k_n}(B - x)|.$$

The estimation of remainder term $R_n^{(2)}(B - x)$ follows from the assertions of Theorem 6. \square

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