

# Information variation during the measurement process

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## Abstract

Quality of results of physical measuring system is analysed mathematically in this publication. Analysis of information amount function shows that information amount on measurement errors obtained from closing error is decreasing when accuracy of measurement is increasing, while calibration errors of the system stay constant. Errors of calibration describe information amount obtained during the time of measurement. Based on curvature of information amount function the optimal measurement accuracy variant is obtained when the ratio of errors of measurement and calibration is equal to three. The impact of tolerance on measurement information amount is demonstrated.

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## 1. Introduction

Quality of physical measuring system could be analysed by amount of information produced during the measurement process. Therefore the data on appropriate system will be more valuable and richer if the system was better defined before the measurement [1–12].

A priori measure of uncertainty of particular measurement system is called entropy and this characteristic is used in informatics theory. Entropy of discrete system  $H(X)$  is described:

$$H(X) = - \sum p_i \log p_i, \quad (1)$$

where  $H(X)$ —symbol of entropy,  $p_i$ —probability of system  $X_i$  state.

Entropy of systems of continuous type (or random value) is expressed:

$$H_{\Delta x}(X) = H'(X) - \log \Delta x = -M\{\log[f(x)\Delta x]\}, \quad (2)$$

where  $H'(X)$ —reduced entropy,  $\Delta x$ —discreteness interval of continuous system,  $M$ —mean value symbol. Therefore only beginning of entropy system computing depends on  $\Delta x$  measurement accuracy.

Entropy  $H'(X)$  is obtained from formula:

$$H'(X) = - \int_{-\infty}^{\infty} f(x) \log f(x) dx, \quad (3)$$

where  $f(x)$ —function of probability density.

Logarithm can be computed but applying any of it's radix  $a > 1$ . Practically entropy is easier

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computed using computers in binary system when radix  $a = 2$ . We will use base of binary system in further computations.

Information amount obtained as measurement results when calibration errors are used in analysis procedures is analysed in this article. Also effect of tolerance to amount of collected information is estimated.

**2. Theoretic principles of information amount determination**

Reference value  $x_e$  of measured value  $X$  and measured result  $x_m$  difference  $\omega$  is kind of closing error which is composed of calibration value and measuring result errors:

$$\omega = x_m - x_e = \Delta_m - \Delta_e, \tag{4}$$

where  $\Delta_m, \Delta_e$ —errors of value  $X$  measuring result and calibration value, composed of random and systematic components, when

$$\Delta_m = \Delta_{m,rand.} + \Delta_{m,syst.}, \quad \Delta_e = \Delta_{e,rand.} + \Delta_{e,syst.}.$$

Closing error  $\omega$  gives mean information amount on measuring result  $x_m$  and its error  $\Delta_m$  can be expressed [1,11]:

$$I_{\omega \rightarrow x_m} = H(x_m) - H(x_m|\omega) = M \left\{ \log \frac{f(x_m|\omega)}{f(x_m)} \right\}, \tag{5}$$

where  $I_{\omega \rightarrow x_m}$ —is a mean information amount on  $x_m$ , which is obtained during the measurement time;  $M$ —mean value (expectation) symbol,  $f(x_m|\omega)$ —random values  $x_m$  conditional probability density, when closing error get value  $\omega$ .

Tolerance effect on collected information amount is not estimated in the above formulas. Tolerance influence on obtained information can be expressed as changed shape of truncated probability density function. Then equality (5) can be written [11]:

$$I_{\omega \rightarrow x_m} = M \left\{ \log \frac{f'(x_m|\omega)}{f'(x_m)} \right\}, \tag{6}$$

where  $f'(x_m), f'(x_m|\omega)$ —truncated probability densities.

Other expression of formula (6) is

$$I_{\omega \rightarrow x_m} = \int_{-d}^d f'(x_m) \log f'(x_m) dx_m - \int_{-d}^d f'(x_m|\omega) \log f'(x_m|\omega) dx_m. \tag{7}$$

Tolerance  $d$  is recorder depending on accepted probability  $\alpha$  and measurement standard deviation  $\sigma_x$ , that is

$$d = t_\alpha \sigma_x, \tag{8}$$

where  $t_\alpha$ —parameter, which value depends from probability  $\alpha$ ;  $\sigma_x$ —standard deviation.

Truncated probability densities expressed through unconditional probability densities [11]:

$$f'(x_m) = \frac{1}{\alpha_1} f(x_m), \tag{9}$$

$$f'(x_m|\omega) = \frac{1}{\alpha_2} f(x_m|\omega), \tag{10}$$

where  $\alpha_1, \alpha_2$ —corresponding probabilities.

Tolerances of values  $x_m$  and  $x_m|\omega$  of magnitude  $X$  have expressions:

$$d_{x_m} = t_{\alpha_1} \sigma_{x_m}, \tag{11}$$

$$d_{x_m|\omega} = t_{\alpha_2} \sigma_{x_m|\omega}. \tag{12}$$

Further we will use expression

$$f(\omega, x_m) = f(x_m|\omega)f(\omega) = f(\omega|x_m)f(x_m), \tag{13}$$

where  $f(\omega, x_m)$ —common probability density of system  $(\omega, x_m)$ .

We analyse information amount collected during measurement process using two cases, by taking into account and not the tolerance effect on the amount of information collected.

**3. Information amount when tolerance is ignored**

Formula (5), taking into account (13), could be written (tolerance effect is not valued):

$$I_{\omega \rightarrow x_m} = M \left\{ \log \frac{f(\omega|x_m)}{f(\omega)} \right\}. \tag{14}$$

Assumption can be made, that measurement results have normal distribution:

$$f(\omega) = \frac{1}{\sqrt{2\pi}\sigma_\omega} \exp \left\{ -\frac{(\omega - M\omega)^2}{2\sigma_\omega^2} \right\}, \tag{15}$$

$$f(\omega|x_m) = \frac{1}{\sqrt{2\pi}\sigma_e} \exp \left\{ -\frac{[\omega - M(\omega|x_m)]^2}{2\sigma_e^2} \right\}, \tag{16}$$

where  $\sigma_\omega, \sigma_e$ —standard deviations of closing error and reference value  $x_e$  standard deviations,  $\sigma_\omega = \sqrt{\sigma_m^2 + \sigma_e^2}, M(\omega|x_m) = \Delta_m - \Delta_{e,sist.}, \omega - M(\omega|x_m) = -\Delta_{e,ats.}, \sigma_{\omega|x_m} = \sigma_e$ .

Components of formula (14) are

$$M\{\log f(\omega|x_m)\} = -(\log \sqrt{2\pi} + \log \sigma_e) - \frac{\sigma_e^2}{2\sigma_e^2} \log e,$$

$$M\{\log f(\omega)\} = -(\log \sqrt{2\pi} + \log \sigma_\omega) - \frac{\sigma_\omega^2}{2\sigma_\omega^2} \log \omega.$$

Further expressions (14) can be written:

$$I_{\omega \rightarrow x_m} = \log \frac{\sigma_\omega}{\sigma_e} = \log \sqrt{1 + \frac{\sigma_m^2}{\sigma_e^2}}. \tag{17}$$

Information amount  $I_{\omega \rightarrow x_c}$  from closing error  $\omega$  of reference value  $x_c$  and its error  $\Delta_e$  can be computed in the same way:

$$I_{\omega \rightarrow x_c} = \log \frac{\sigma_\omega}{\sigma_m} = \log \sqrt{1 + \frac{\sigma_e^2}{\sigma_m^2}}. \tag{18}$$

Results of information amount computations are presented in Table 1.

Table data are plotted in the graph and showed in Fig. 1.

Function  $I_{\omega \rightarrow x_m}$  graph shows that information amount on observation quantities errors determined from closing error is increasing when accuracy of observations is decreasing. Calibration errors describe amount of information, which is not increasing when accuracy of observations is increasing. Therefore accuracy of observed magnitudes and instruments calibration errors should be matching.

We analyse optimal accuracy case of measurements. Points with extreme curvature of function  $I_{\omega \rightarrow x_m}$  will be used. Information amount changes significantly in these points and is increasing in the case of analysed function when argument  $\varepsilon = \sigma_m/\sigma_e$  is increasing.

Table 1  
Information amount when tolerance is ignored

$\varepsilon = \frac{\sigma_m}{\sigma_e}$	Information amount in binary units	
	$I_{\omega \rightarrow x_m}$	$I_{\omega \rightarrow x_c}$
10	3.329	0.007
5	2.350	0.028
3.3	1.786	0.063
2	1.161	0.161
1	0.500	0.500
0.5	0.161	1.161
0.33	0.075	1.674
0.25	0.044	2.043
0.2	0.028	2.350
0.1	0.007	3.329

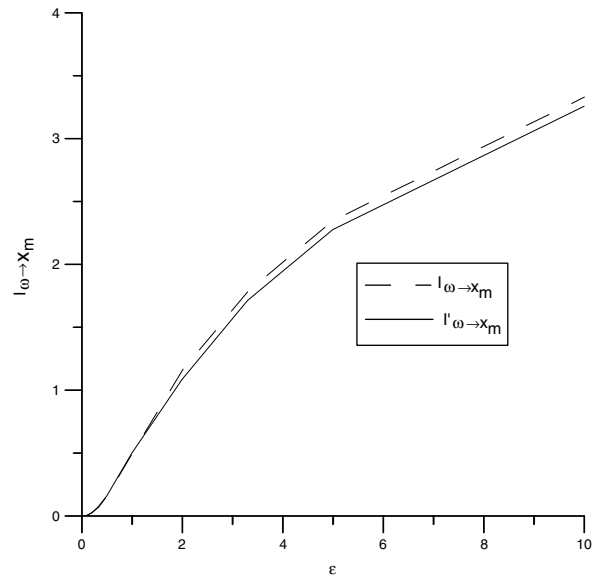


Fig. 1. Graph of information amount variation.

Function  $I_{\omega \rightarrow x_m}$  curve equation is

$$K = \frac{\frac{d^2 I}{d\varepsilon^2}}{\left[1 + \left(\frac{dI}{d\varepsilon}\right)^2\right]^{3/2}}, \tag{19}$$

where  $K$ —curve equation,  $\varepsilon = \sigma_m/\sigma_e$ —new variable.

Curve equation  $K$  extreme points are determined by solving equation

$$\frac{dK}{d\varepsilon} = 0.$$

Roots of solution are equal:  $\varepsilon_{1,2} = \pm 2.8$ .

Therefore magnitude measurement standard deviation  $\sigma_m$  should be approximately three times larger than instrument calibration standard deviation  $\sigma_e$ .

#### 4. Information amount when tolerance is taken into account

We will analyse information amount expression  $I'_{\omega \rightarrow x_m}$  taking into account tolerance:

$$I'_{\omega \rightarrow x_m} = M\left\{\log \frac{f'(\omega|x_m)}{f'(\omega)}\right\} = M\left\{\log \frac{\alpha_1 f(\omega|x_m)}{\alpha_2 f(\omega)}\right\}. \tag{20}$$

Further on

$$d_\omega = t_{z_1} \sigma_\omega,$$

$$d_{\omega|x_m} = t_{z_2} \sigma_{\omega|x_m} = t_{z_2} \sigma_e.$$

Table 2  
Information amount when tolerance is estimated

$\varepsilon = \frac{\sigma_m}{\sigma_e}$	Information amount in binary units	
	$t_{z_1} = 2 \ (\alpha_1 = 0.9545)$	
	$I'_{\omega \rightarrow x_m}$	$I'_{\omega \rightarrow x_e}$
10	3.256	0.007
5	2.277	0.028
3.3	1.713	0.063
2	1.088	0.161
1	0.500	0.500
0.5	0.161	1.088
0.33	0.075	1.601
0.25	0.044	1.970
0.2	0.028	2.277
0.1	0.007	3.256

Because  $d_\omega = d_{\omega|x_m}$ , then  $t_{z_1} \sigma_\omega = t_{z_2} \sigma_e$  and

$$t_{z_1} = t_{z_2} \frac{\sigma_e}{\sigma_\omega} = t_{z_2} \frac{\sigma_e}{\sqrt{\sigma_m^2 + \sigma_e^2}} = t_{z_2} \frac{1}{\sqrt{1 + \frac{\sigma_m^2}{\sigma_e^2}}}. \quad (21)$$

By applying tables of P.S. Laplace function in the case of normal distribution when probability  $\alpha_1$  computed parameter  $t_{z_1}$  value and further on from formula (21)  $t_{z_2}$  is computed, as well as  $\alpha_2$ .

Information amount expression  $I'_{\omega \rightarrow x_m}$  from formula (20) and using Eq. (17), taking into account tolerance, can be written:

$$I'_{\omega \rightarrow x_m} = \log \left\{ \frac{\alpha_1}{\alpha_2} \frac{\sigma_\omega}{\sigma_e} \right\} = \log \left\{ \frac{\alpha_1}{\alpha_2} \sqrt{1 + \frac{\sigma_m^2}{\sigma_e^2}} \right\}. \quad (22)$$

Information amount  $I'_{\omega \rightarrow x_e}$ , derived from closing error  $\omega$  on reference magnitude  $x_e$ , taking into account tolerance, is expressed:

$$I'_{\omega \rightarrow x_e} = \log \left\{ \frac{\alpha_1}{\alpha_2} \frac{\sigma_\omega}{\sigma_m} \right\} = \log \left\{ \frac{\alpha_1}{\alpha_2} \sqrt{1 + \frac{\sigma_e^2}{\sigma_m^2}} \right\}. \quad (23)$$

We compute information amounts  $I'_{\omega \rightarrow x_m}$  and  $I'_{\omega \rightarrow x_e}$  from formulas (22) and (23) and taking into account tolerance, when observation results have a normal distribution and  $t_{z_1} = 2 \ (\alpha_1 = 0.9545)$ . Computation results are presented in Table 2 and showed in Fig. 1.

### 5. Conclusions

1. Calibration errors of measuring instruments describe information amount, which decrease when accuracy of particular measuring instrument is increasing. This shows that measurement accuracy of appropriate instrument and its calibration errors should be matching.
2. Optimal standard deviation of measurements is approximately three times larger than instrument calibration standard deviation.
3. Tolerance influence on measurement information amount is not big and reaches approximately 5%.

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