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# MAXIMIZING AND MINIMIZING SETS IN SOLVING FUZZY LINEAR PROGRAMMING

***Abstract.*** *Linear programming with fuzzy information is a continuous field of researches in uncertain programming. Since the lack of a certain and deterministic solution is a natural characteristic of problems under uncertainty, different methods proposed various schemes to solve such problems. In this paper, a new framework is developed to solve fuzzy linear programming where the problem's parameters, include objective function coefficients, technological matrix elements and right hand side values, are stated as fuzzy numbers. The proposed method is based on the notion of maximizing and minimizing sets, as a well known and widely accepted method of fuzzy numbers ranking, and tries to find a solution which optimizes the utility function of fuzzy objective functions by considering fuzzy constraints which are analyzed based on the concept of α-cuts and interval numbers relations. To show the applicability of the proposed method, its application is illustrated in a numerical example and its results are compared with a current method.*

***Keywords:*** *Fuzzy linear programming; fuzzy numbers; minimization set; maximization set.*

**JEL Classificalion: CO2**

## 1 Introduction

An inherent feature of all human being and physical/natural systems is optimization due to limitation of available resources (Nocedal and Wright, 1999). In general, an optimization problem can be defined as finding infimum or supremum of a given real-valued function *f* over a specified set *G* of a universal set *X*. i.e.

$$\alpha = \inf \{f(x) : x \in G\}, G \subseteq X \tag{1}$$

The optimization problem includes finding the value of $\alpha$ or equivalently, $x_0 \in G$ that $f(x_0) = \alpha$ (Ponstein, 2004). Optimization methods can be classified under exact and approximate methods (Talibi, 2009). The original linear programming problem, introduced by Dantzig (1948) is an exact optimization problem which is defined as optimization (i.e. minimization or maximization) of a linear function while a set of linear constraints are satisfied. Mathematically it can be stated as follows:

$$\begin{aligned} &\text{Max}\, CX \\ &\text{Subject to} \\ &AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0 \end{aligned} \tag{2}$$

Where X is $(n \times 1)$ column vector of decision variables, C is $(1 \times m)$ row vector of profit (cost) coefficients, A denotes $(m \times n)$ technological matrix, and b represents $(m \times 1)$ resources or right hand side vector (Bazaraa et al. 2009). A comprehensive review on original linear programming concepts and techniques can be found in Murthy (1983) and Bazaraa et al. (2009).

In conventional linear programming problems, the elements of matrix A and also vectors C and b are considered as deterministic numbers. However, practical problems are usually engaged with a subject of uncertainty. Uncertainty may occur due to one of the following reasons: (1) the information is partial, or (2) the information is approximate (Yovits, 1984). Different frameworks are introduced in response to modeling and analyzing of the uncertain systems. Fuzzy set theory (Zadeh, 1965) is one of the most well known and widely accepted frameworks regarding the uncertainty. Fuzzy sets are generalized form of classic sets in which a membership degree (function) is assigned to each element of a universal set. Bellman and Zadeh (1970) initially introduced the notion of fuzzy decision making which was widely developed later by researchers.

Linear programming problems are widely examined under fuzziness of their data. Fuzzy linear programming (FLP) with fuzzy coefficients was formulated by Negoita (1970) and followed by Zimmermann (1983), and Tanaka and Asai (1984). Since then, work on FLP grew continuously. Chanas (1983) developed a technique based on parametric programming to solve the FLP problems. Delgado et al. (1989) developed a general framework for solving the FLP problems. In another research work, Shaocheng (1994) applied two approaches based on fuzzy decisive set and interval linear programming for several membership levels to solve FLP problems. Guua and Wu (1999) proposed a two-phase approach to solve FLP problems, while Jamison and Lodwick (2001) transformed the FLP problem into an unconstrained programming problem by penalizing possible violation from the constraints. They completed their work by optimizing this function which was proved to be a concave function. Yazdani Peraei et al. (2001) suggested a method by the means of comparing the fuzzy numbers and introduced a method for solving the FLP problem based on Mellin transformation. Jiménez et al. (2007) developed a method for solving the FLP by using a fuzzy ranking method. Mahdavi-Amiri and Nasseri (2007) applied a linear ranking function to order trapezoidal fuzzy numbers, extended the concepts of duality to FLP and introduced a dual algorithm for solving the FLP. Zangiabadi and Maleki (2007) transformed a fuzzy linear programming problem to a multi-objective linear programming problem. Kaur and Kumar (2013) introduced a new method based on Farhadinia's ranking approach (2009) to find a unique fuzzy optimal solution for the FLP problems.

Beside the theoretical approaches which are developed for solving FLP, like the above mentioned cases, the FLP has wide applications. Among these applications, one can refer to the applications of FLP in multi criteria decision making (Li and Sun, 2007, Li and Yang, 2004), production planning (Vasant, 2003; Vasant et al., 2004, Wang and Zheng, 2013), reliability assessment (Verma et al., 2005), project management (Liang, 2006), distribution planning (Bilgen and Ozkarahan, 2006), transportation planning (Liang, 2008) design of new product (Chen and Ko, 2009), waste management (Fan et al., 2009), supply chain planning (Bilgen, 2010, Peidro et al., 2010), supplier selection and order allocation (Amin et al., 2011), irrigation planning (Regulwar and Gurav, 2011), environmental management (Fan et al., 2012), and etc. These contributions along with lack of a deterministic solution for the uncertain problems, as noted by Liu and Lin (2006), keep the FLP problem an ongoing field of research.

In this paper, a new approach is developed to solve the fuzzy linear programming problems based on maximization and minimization sets of a fuzzy set. The aim of this approach is to construct the maximization and minimization sets of the fuzzy linear objective function. Therefore, it tries to find a feasible point, in the sense of fuzziness, with greatest possible value in the maximization set and simultaneously the lowest possible value in the minimization set. This paper is organized as follows. A brief overview of fuzzy set theory and required concepts is given in section 2. The fuzzy linear programming problem and the required definitions are provided in section 3. Section 4 explains the proposed approach. In section 5, an FLP problem is solved with the proposed method and its results are compared with Jiménez et al. (2007). Also, an application is inspired from a real world problem and is solved with the presented method. Finally, section 6 makes some conclusions.

## 2 Fuzzy Sets

Fuzzy sets are introduced by Zadeh (1965) as a generalized form of classic sets. Suppose that $U$ is a universe. A fuzzy set $\tilde{A}$ in $U$ is defined as $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in U\}$, where $\mu_{\tilde{A}}(x)$ is called the membership function of $\tilde{A}$. If $\mu_{\tilde{A}}(x): U \rightarrow [0,1]$, then $\tilde{A}$ is called a normal fuzzy set. Jain (1976, 1977) plus Dubois and Prade (1978) defined the concept of fuzzy numbers for the first time. A fuzzy number is a normal and convex fuzzy set $\tilde{A}$ in the universe $U$. The most common form of the fuzzy numbers in practical problems, especially in problems which are related to decision making are trapezoidal and triangular fuzzy numbers. This feature is usually attributed to their good practicability and ease of understanding. A trapezoidal fuzzy number can be shown as the quadruple $\tilde{A} = (l, m_1, m_2, r)$, where $l \le m_1 \le m_2 \le r$ are real numbers, while a trapezoidal fuzzy number $\tilde{A}$ is characterized by its membership functioning as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le l \\ \dfrac{x-l}{m_1-l}, & l \le x \le m_1 \\ 1, & m_1 \le x \le m_2 \\ \dfrac{r-x}{r-m_2}, & m_2 \le x \le r \\ 0, & x \ge r \end{cases} \tag{3}$$

Triangular fuzzy numbers are a specific form of trapezoidal fuzzy numbers, where $m_1 = m_2$. Let $\tilde{A} = (l_1, m_{11}, m_{12}, r_1)$ and $\tilde{B} = (l_2, m_{21}, m_{22}, r_2)$ be two trapezoidal fuzzy numbers. The algebraic operations on these numbers can be defined as follows (Dubois and Prade, 1980, Rommelfanger, 1994, 1996):

$$\tilde{A} \oplus \tilde{B} = (l_1 + l_2, m_{11} + m_{21}, m_{12} + m_{22}, r_1 + r_2) \tag{4}$$

$$-\tilde{A} = (-r_1, -m_{12}, -m_{11}, -l_1) \tag{5}$$

$$\tilde{A} - \tilde{B} = (l_1 - r_2, m_{11} - m_{22}, m_{12} - m_{21}, r_1 - l_2) \tag{6}$$

$$\tilde{A} \otimes \tilde{B} \approx (l_1 \cdot l_2, m_{11} \cdot m_{21}, m_{12} \cdot m_{22}, r_1 \cdot r_2) \tag{7}$$

It should be noted that multiplication of the trapezoidal fuzzy numbers doesn't result in the same type of fuzzy numbers and Eq. (5) is just an approximation of the result.

For a fuzzy set $\tilde{A}$, its $\alpha$-cut is defined as $\tilde{A} = \{x \in U \mid \mu_{\tilde{A}}(x) \ge \alpha\}$.

These $\alpha$-cuts can be shown as crisp intervals which are also called $\alpha$-level intervals:

$$(\tilde{A})_\alpha = \left[(\tilde{A})_\alpha^l, (\tilde{A})_\alpha^u\right] = \left[\min_x \{x \in U \mid \mu_{\tilde{A}}(x) \ge \alpha\}, \max_x \{x \in U \mid \mu_{\tilde{A}}(x) \ge \alpha\}\right] \tag{8}$$

For a trapezoidal fuzzy number $\tilde{A} = (l, m_1, m_2, r)$, its $\alpha$-level interval is determined as follows:

$$(\tilde{A})_\alpha = [m_1\alpha + l(1-\alpha), m_2\alpha + r(1-\alpha)] \tag{9}$$

An important concept to develop the FLP algorithm in this paper are maximizing set and minimizing set which are introduced by Chen (1985) to rank a set of fuzzy numbers. Assume that $\tilde{A}_i, i = 1,2,\ldots,n$ are a set of $n$ fuzzy numbers with membership functions $f_{\tilde{A}_i}, i = 1,2,\ldots,n$ which are defined on universal sets $S_i, i = 1,2,\ldots,n$. The maximizing set $M$ and minimizing set $G$ with the membership functions $f_M$ and $f_G$ are determined as follows.

$$f_M(x) = \begin{cases} \dfrac{(x - x_{\min})}{(x_{\max} - x_{\min})}, & x_{\min} \le x \le x_{\max} \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$$f_G(x) = \begin{cases} \dfrac{(x_{\max} - x)}{(x_{\max} - x_{\min})}, & x_{\min} \le x \le x_{\max} \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Where, $x_{\min} = \text{Inf}\, S$, $x_{\max} = \text{Sup}\, S$, $S = \bigcup_{i=1}^{n} S_i$, and $S_i = \{x \mid f_{\tilde{A}_i}(x) > 0\}$. Chou et al. (2011) proposed a revised version of ranking fuzzy numbers using the maximization and minimization sets. If $f_{\tilde{A}_i}^L(x)$ and $f_{\tilde{A}_i}^R(x)$ are the left and right membership functions of $\tilde{A}_i$, respectively, then the right utility of $\tilde{A}_i$ is defined as:

$$UM_{i1}(x) = \text{Sup}_x \left(f_M(x) \wedge f_{\tilde{A}_i}^R(x)\right) \tag{12}$$

$$UG_{i2}(i) = \operatorname{Sup}_x \left( f_G(x) \wedge f^R_{\tilde{A}_i}(x) \right) \tag{13}$$

And the left utility of $\tilde{A}_i$ is defined as:

$$UG_{i1}(i) = \operatorname{Sup}_x \left( f_G(x) \wedge f^R_{\tilde{A}_i}(x) \right) \tag{14}$$

$$UM_{i2}(i) = \operatorname{Sup}_x \left( f_M(x) \wedge f^L_{\tilde{A}_i}(x) \right) \tag{15}$$

Finally, the total utility of $\tilde{A}_i$ with an optimality index $\alpha$ is calculated as below:

$$U_T^{\alpha}(i) = \left\{ \begin{array}{l} \alpha \left[ UM_{i1}(i) + 1 - UG_{i2}(i) \right] \\ + (1-\alpha) \left[ UM_{i2}(i) + 1 - UG_{i1}(i) \right] \end{array} \right\} \tag{16}$$

The optimality index $\alpha$ shows the degree of decision maker's optimistic viewpoint. If $\alpha = 0$, $U_T^0(i)$ shows a pessimistic decision maker while for $\alpha = 1$, the $U_T^1(i)$ shows an optimistic one. For a realistic decision maker, $\alpha = 1/2$ . However, for a trapezoidal fuzzy number $\tilde{A}_i = (l_i, m_{1i}, m_{2i}, r_i)$ and a realistic decision maker, $U_T^{1/2}(i)$ is defined as below:

$$U_T^{1/2}(i) = \frac{1}{2} \left\{ \left[ \frac{r_i - x_{\min}}{r_i - m_{2i} + x_{\max} - x_{\min}} + \frac{m_{2i} - x_{\min}}{m_{2i} - r_i + x_{\max} - x_{\min}} \right] + \left[ \frac{l_i - x_{\min}}{l_i - m_{1i} + x_{\max} - x_{\min}} + \frac{m_{1i} - x_{\min}}{m_{1i} - l_i + x_{\max} - x_{\min}} \right] \right\} \tag{17}$$

## 3 Fuzzy Linear Programming

As stated in section 1, an FLP problem is a linear programming problem, according to Eq. (2), where at least one element of its parameters including vectors *c* and/or *b* and/or matrix *A* is represented in the form of fuzzy numbers. Thus, a fuzzy linear programming can be shown as follows:

$$
\begin{aligned}
&\text{Max}\,\tilde{C}X \\
&\text{Subject to} \\
&\tilde{A}X \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} \tilde{b},\ X \ge 0
\end{aligned}
\tag{18}
$$

Where $X$ is $(n \times 1)$ column vector of the decision variables, $\tilde{C}$ is $(1 \times m)$ fuzzy row vector of profit (cost) coefficients, $\tilde{A}$ denotes $(m \times n)$ fuzzy technological matrix, while $\tilde{b}$ gives $(m \times 1)$ fuzzy resources or right hand side vector. In an extended form, Eq. (18) can be written in the following form:

$$
\begin{aligned}
&\max \sum_{j=1}^{n} \tilde{c}_j x_j \\
&S.T. \\
&\sum_{j=1}^{n} \tilde{a}_{ij} x_j \begin{pmatrix} \le \\ = \\ \ge \end{pmatrix} \tilde{b}_i,\ i = 1,2,\ldots,m \\
&x_j \ge 0,\ j = 1,2,\ldots,n
\end{aligned}
\tag{19}
$$

Since the trapezoidal fuzzy numbers are a prominent shape of fuzzy numbers in decision making problems, therefore in the current paper, the parameters are considered as the trapezoidal numbers. Therefore,

$$
\begin{aligned}
&\tilde{c}_j = (c_{1j}, c_{2j}, c_{3j}, c_{4j}),\ j = 1,2,\ldots,n \\
&\tilde{a}_{ij} = (a_{1ij}, a_{2ij}, a_{3ij}, a_{4ij}),\ i = 1,2,\ldots, \\
&m;\ j = 1,2,\ldots,n,\ \tilde{a}_{ij} \in \tilde{A} \\
&\tilde{b}_i = (b_{1i}, b_{2i}, b_{3i}, b_{4i}),\ i = 1,2,\ldots,m
\end{aligned}
$$

Meanwhile, the algebraic operations in Eq. (19) on the trapezoidal fuzzy numbers are done based on Eqs. (4) to (7).

## 4 FLP Solving Approach

There is a wide variety of procedures to solve the FLP problems. In this section, a method is proposed based on the maximization and minimization sets.

### 4.1 Constraints

To deal with the constraints of Eq. (19), an approach based on $\alpha$-cut is adopted. Consider $i^{th}$ constraint of the problem. The $\tilde{a}_{ij}$ coefficient is transformed into its equivalent $\alpha$-cut according to Eq. (9) as follows:

$$(\tilde{a}_{ij})_{\alpha} = [a_{2ij}\alpha + a_{1ij}(1-\alpha), a_{3ij}\alpha + a_{4ij}(1-\alpha)] \tag{20}$$

Applying this transformation into all coefficients of this constraint and summing them over $j$ indices, the constraint will become as:

$$\left[\sum_{j=1}^{n} (a_{2ij}\alpha + a_{1ij}(1-\alpha))x_j, \sum_{j=1}^{n} (a_{3ij}\alpha + a_{4ij}(1-\alpha))x_j\right] \le [b_{2i}\alpha + b_{1i}(1-\alpha), b_{3i}\alpha + b_{4i}(1-\alpha)] \tag{21}$$

In order to compare the left hand side and right hand side of Eq. (21), some order relations can be used, as those introduced by Ishibuichi and Tanaka (1990). If $A_1 = [\underline{A}_1, \overline{A}_1]$ and $A_2 = [\underline{A}_2, \overline{A}_2]$ are two interval numbers, then the following order relations will be defined as below:

-If $[\underline{A}_1, \overline{A}_1] \le [\underline{A}_2, \overline{A}_2]$, then according to the order relation $\le_{RC}^{*}$, it is expected that:

$$\begin{cases} \overline{A}_1 \le \overline{A}_2 \\ \left(\dfrac{\overline{A}_1 + \underline{A}_1}{2}\right) \le \left(\dfrac{\overline{A}_2 + \underline{A}_2}{2}\right) \end{cases} \tag{22}$$

If $[\underline{A}_1, \overline{A}_1] \ge [\underline{A}_2, \overline{A}_2]$, then according to the order relation $\le_{LC}^{*}$, it follows that:

$$\begin{cases} \underline{A}_1 \ge \underline{A}_2 \\ \left(\dfrac{\overline{A}_1 + \underline{A}_1}{2}\right) \ge \left(\dfrac{\overline{A}_2 + \underline{A}_2}{2}\right) \end{cases} \tag{23}$$

$[\underline{A}_1, \overline{A}_1] = [\underline{A}_2, \overline{A}_2]$ if and only if $\underline{A}_1 = \underline{A}_2$ and $\overline{A}_1 = \overline{A}_2$.

#### 4.2 Objective Function

Consider the objective function in Eq. (19): If the multiplication and summation operations are done on the objective function, the result will be a trapezoidal fuzzy number based on Eqs. (4) to (7). Suppose that this number has such a form as follows:

$$\max \tilde{Z} = (C_1x, C_2x, C_3x, C_4x) \tag{24}$$

Where $C_4x \geq C_3x \geq C_2x \geq C_1x$ . According to Eq. (17), total utility of the objective function, $U_T^{1/2}(\tilde{Z})$ can be defined as follows:

$$\begin{aligned} U_T^{1/2}(\tilde{Z}) = \frac{1}{2}\Bigg\{ & \left[ \frac{C_4x - Z_{\min}}{C_4x - C_3x + Z_{\max} - Z_{\min}} + \frac{C_3x - Z_{\min}}{C_3x - C_4x + Z_{\max} - Z_{\min}} \right] \\ & + \left[ \frac{C_1x - Z_{\min}}{C_1x - C_2x + Z_{\max} - Z_{\min}} + \frac{C_2x - Z_{\min}}{C_2x - C_1x + Z_{\max} - Z_{\min}} \right] \Bigg\} \end{aligned} \tag{25}$$

To construct the $U_T^{1/2}(\tilde{Z})$, $Z_{\max}$ and $Z_{\min}$ must be specified first. These values are determined by solving Eqs. (26) and (27):

$$\begin{aligned} & Z_{\max} = \max C_4 x \\ & \text{Subject To} \\ & x \in S_0 \end{aligned} \tag{26}$$

Similarly,

$$\begin{aligned} & Z_{\min} = \max C_1 x \\ & \text{Subject To} \\ & x \in S_1 \end{aligned} \tag{27}$$

Now, the $U_T^{1/2}(\tilde{Z})$ can be constructed and the model (19) in a given α-level is obtained as below:

$$
\begin{aligned}
& Z_{\alpha} = \max U_T^{1/2}(\tilde{Z}) \\
& \text{Subject To} \\
& x \in S_{\alpha}
\end{aligned}
\tag{28}
$$

Model (28) is a fractional programming problem which is solved based on the method proposed by Dutta et al. (1992). His model is based on maximizing the summed membership functions of nominators and denominators. Considering the first fraction in Eq. (25), the membership function of its nominator would be defined as follows:

$$
\mu_1^N = \begin{cases} 0, & \text{if } (C_4 x - Z_{\min}) \leq N_1^- \\ \dfrac{(C_4 x - Z_{\min}) - N_1^-}{N_1^* - N_1^-}, & \text{if } N_1^- \leq (C_4 x - Z_{\min}) \leq N_1^* \end{cases}
\tag{29}
$$

Where,

$$
\begin{aligned}
& N_1^- = \max C_4 x - Z_{\min} \\
& \text{Subject To} \\
& x \in S_1
\end{aligned}
\tag{30}
$$

And

$$
\begin{aligned}
& N_1^* = \max C_4 x - Z_{\min} \\
& \text{Subject To} \\
& x \in S_0
\end{aligned}
\tag{31}
$$

Similarly, for the denominator of the first fraction it holds that

$$
\mu_1^D = \begin{cases} 0, \ \text{if } (C_4 x - C_3 x + Z_{\max} - Z_{\min}) \geq D_1^- \\ \dfrac{D_1^- - (C_4 x - C_3 x + Z_{\max} - Z_{\min})}{D_1^- - D_1^*}, \\ \quad \text{if } D_1^* \leq (C_4 x - C_3 x + Z_{\max} - Z_{\min}) \leq D_1^- \end{cases}
\tag{32}
$$

Where,

$$
\begin{aligned}
& D_1^- = \min C_4 x - C_3 x + Z_{\max} - Z_{\min} \\
& \text{Subject To} \\
& x \in S_1
\end{aligned}
\tag{33}
$$

And

$$D_1^* = \min C_4 x - C_3 x + Z_{\max} - Z_{\min}$$
$$\text{Subject To} \tag{34}$$
$$x \in S_0$$

If one of the elements of $N_i^-$, $N_i^*$, $D_i^-$, and $D_i^*$ has an infinite optimal solution, it can be considered as equal to a great real value. e.g. $M$.

These calculations are done for all four fractions of Eq. (25). Finally the objective function is found as follows:

$$Z = \sum_{i=1}^{4} \left(\mu_i^N + \mu_i^D\right) \tag{35}$$

And therefore, the problem (28) would be as follows:

$$Z_\alpha = \max Z$$
$$\text{Subject To} \tag{36}$$
$$x \in S_\alpha$$

### 4.3 Fuzzy Linear Programming Algorithm based on Minimization and Maximization Sets

In sections 4-1 and 4-2, the ways of handling the constraints and objective function of a typical FLP problem are described (Eq. (19)). In this sub-section, the proposed FLP algorithm is explained based on the minimization and maximization sets:

1) Transform the original constraints of FLP into their equivalent forms in Eq. (23);
2) Determine $Z_{\max}$ and $Z_{\min}$ by solving Eqs. (26) and (27), respectively;
3) Determine the values of $N_i^-$, $N_i^*$, $D_i^-$, and $D_i^*$ by solving Eqs. (30), (31), (33) and (34), respectively.
4) Solve the model (36) for different levels of $\alpha$;
5) Determine the final optimal value.

Since the problem is solved for different values of $\alpha$, a set of optimal solutions will be available. Suppose that the problem is solved for $n$ values of $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. Two factors must be considered to choose the final solution: 1) its membership in maximization set (Eq. (10)) and 2) how it satisfies the feasible space. To extract the first factor, Yager's model (1979) is used:

$$K_{f_M}(Z_\alpha)=\frac{\int_{-\infty}^{+\infty}\mu_{\tilde{Z}_\alpha}(z)\cdot\mu_{f_M}(z)\,dz}{\int_{-\infty}^{+\infty}\mu_{\tilde{Z}_\alpha}(z)\,dz} \tag{37}$$

When the objective function is of minimization type, $f_M(x)$ is replaced by $f_G(x)$. Then, this degree is justified by the level of α:

$$M_{Z_\alpha}=K_{f_M}(Z_\alpha)\cdot\alpha \tag{38}$$

Finally, the best solution is specified with the highest degree:

$$x^*=\max_{i=1,2,\ldots,n} M_{Z_{\alpha_i}} \tag{39}$$

**Note**. Sometimes it is possible that: (1) $N_i^-=N_i^*$ either $D_i^-=D_i^*$ or, (2) the optimal solution for one of the problems associated with $N_i^-, N_i^*, D_i^-$, or $D_i^*$ is infinite. In these cases:

(1) To avoid the denominators of Eq. (25) becoming zero, a deviation factor $0<\Delta\le 1$ is added to $N_i^-$ or it is subtracted from $D_i^-$, so that $N_i^*=(1+\Delta)N_i^-$ or $D_i^*=(1-\Delta)D_i^-$. This deviation factor presented the acceptable range of deviation from ideal points followed by a reduction in the membership degrees.

(2) In the case of infinity, a number (*M*) which is large in comparison with the other optimal values is allocated to the infinite $N_i^-, N_i^*, D_i^-$, or $D_i^*$.

### 4.4 Illustrative Example

To better explain application of the proposed method, a numerical example is considered in this section as compared to another alternative method. Jiménez et al. (2007) solved the following FLP through this procedure:

$$\begin{aligned}
&\min\,(19,20,21)x_1+(29,30,31)x_2\\
&\text{S.T.}\\
&(4.5,5,5.5)x_1+(2.5,3,4)x_2\ge(194,200,206)\\
&(3,4,5)x_1+(6.5,7,7.5)x_2\ge(230,240,250)\\
&x_1\ge 0, x_2\ge 0
\end{aligned} \tag{40}$$

Constraints of the model are transformed first based on Eq. (23). Note that while the coefficients are triangular fuzzy numbers, then in all relations $m_1 = m_2$.

$$
\begin{array}{l}
\left[(4.5+0.5\alpha)x_1 + (2.5+0.5\alpha)x_2,\ (5.5-0.5\alpha)x_1 + (4-\alpha)x_2\right] \geq \left[194+6\alpha, 206-6\alpha\right] \\
\left[(3+\alpha)x_1 + (6.5+0.5\alpha)x_2,\ (5-\alpha)x_1 + (7.5-0.5\alpha)x_2\right] \geq \left[230+10\alpha, 250-10\alpha\right] \\
x_1 \geq 0, x_1 \geq 0
\end{array} \tag{41}
$$

Moreover, the objective function has the following form:

$$
\begin{array}{l}
\min (19x_1 + 29x_2, 20x_1 + 30x_2, 21x_1 + 31x_2) \rightarrow \\
\max (-21x_1 - 31x_2, -20x_1 - 30x_2, -19x_1 - 29x_2)
\end{array} \tag{42}
$$

Now, the values of $Z_{max}$ and $Z_{min}$ are determined.

$$
\begin{array}{l}
Z_{max} : \max -19x_1 - 29x_2 \\
\text{Subject To} \\
5x_1 + 3x_2 \geq 200 \\
4x_1 + 7x_2 \geq 240 \\
x_1 \geq 0, x_1 \geq 0
\end{array} \tag{43}
$$

Which its optimal solution is $Z_{max} = -1066087$ and

$$
\begin{array}{l}
Z_{min} : \max -21x_1 - 31x_2 \\
\text{Subject To} \\
4.5x_1 + 2.5x_2 \geq 196 \\
5x_1 + 3.25x_2 \geq 200 \\
3x_1 + 6.5x_2 \geq 200 \\
4x_1 + 7x_2 \geq 240 \\
x_1 \geq 0, x_1 \geq 0
\end{array} \tag{44}
$$

Which its optimal solution is $Z_{\min} = -1188$. In the next step, the values of $N_i^-$, $N_i^*$, $D_i^-$, and $D_i^*$ are derived. It is notable that in Eqs. (24) and (25), $C_2 = C_3$. Therefore,

$N_1^- = 99.21839$, $N_1^* = 121.9130$, $D_1^- = 171.5222$, and $D_1^* = 168.8695$

$N_2^- = 49.6092$, $N_2^* = 74.95652$, $D_2^- = \mathrm{M}$, and $D_2^* = \mathrm{M} - \varepsilon$

$N_3^- = 0$, $N_3^* = 28$, $D_3^- = \mathrm{M}$, and $D_3^* = \mathrm{M} - \varepsilon$

$N_4^- = 49.6092$, $N_4^* = 74.95652$, $D_4^- = 171.5222$, and $D_4^* = 168.8695$

Now, the objective function, Eq. (35), is constructed and solved. Table1 lists the solutions obtained with the proposed method and the results of Jiménez et al. (2007) in seven different levels of *α*.

**Table 1. Solutions of the proposed method for different levels of *α***

| α-level | Method | Decision vector | Fuzzy objective value |
|---|---|---|---|
| 0.4 | Proposed method | $x_1 = 32.89, x_2 = 15.49$ | (1074.15, 1122.53, 1170.91) |
| | Jiménez et al. (2007) | $x_1 = 28.51, x_2 = 17.32$ | (1043.97, 1089.80, 1135.63) |
| 0.5 | Proposed method | $x_1 = 32.31, x_2 = 15.82$ | (1072.76, 1120.90, 1169.03) |
| | Jiménez et al. (2007) | $x_1 = 28.89, x_2 = 17.78$ | (1064.53, 1111.20, 1157.87) |
| 0.6 | Proposed method | $x_1 = 31.75, x_2 = 16.14$ | (1071.39, 1119.29, 1167.18) |
| | Jiménez et al. (2007) | $x_1 = 29.28, x_2 = 18.24$ | (1085.28, 1132.80, 1159.32) |
| 0.7 | Proposed method | $x_1 = 31.19, x_2 = 16.46$ | (1069.95, 1117.60, 1165.25) |
| | Jiménez et al. (2007) | $x_1 = 29.70, x_2 = 18.72$ | (1107.18, 1155.60, 1204.02) |
| 0.8 | Proposed method | $x_1 = 30.64, x_2 = 16.77$ | (1068.49, 1115.90, 1163.31) |
| | Jiménez et al. (2007) | $x_1 = 30.13, x_2 = 19.20$ | (1129.27, 1178.60, 1227.93) |
| 0.9 | Proposed method | $x_1 = 30.10, x_2 = 17.08$ | (1067.22, 1114.40, 1161.58) |
| | Jiménez et al. (2007) | $x_1 = 30.58, x_2 = 17.70$ | (1152.32, 1202.60, 1252.88) |
| 1.0 | Proposed method | $x_1 = 29.56, x_2 = 17.39$ | (1065.95, 1112.9, 1159.85) |
| | Jiménez et al. (2007) | $x_1 = 31.04, x_2 = 20.20$ | (1175.56, 1226.80, 1278.04) |

To choose the best solution based on Eq. (37), this minimization set will be used:

$$f_G(x) = \begin{cases} \left(\frac{1188 - z}{121.913}\right), & 1066.078 \leq x \leq 1188 \\ 0, & \text{otherwise} \end{cases} \tag{45}$$

The preferred solution for this example is (1065.95, 1112.9, 1159.85) with $x_1 = 29.56$ and $x_2 = 17.39$. It is obvious that for all levels, the solution of proposed method has outperformed the one suggested by Jiménez et al. (2007).

## 5 Application

In this section, the proposed method is examined in a production planning problem. Iran Foolad Kavir (IFK) is a steel and rolling company, which produces 9 different types of fittings from 8 mm to 24. Three different types of bullions are melted for manufacturing process. At the first stage, bullions are melted by smelter machines, before the rolling machines start to produce specific sizes of fittings regarding the production schedule. Since civil projects in Iran, demand for each type of the products is guaranteed. So the main objective is to reach the expected monthly production for each type of the fitting. Information related to coefficients of each product from bullions, capacity of each machine, needed process time as well as expected profit of each final product are summarized in Table 2. Some of the information in this table is stated in the form of fuzzy numbers. Since there is no certainty about the market, the expected profit of the products are represented as triangular fuzzy numbers (*a*, *b*, *c*), where *a* is a pessimistic approximation, *b* denotes most likely and *c* shows optimistic approximation. Moreover, due to the external factors and the differences between the materials, use of the smelter machines and rolling machines are given as triangular fuzzy numbers.

**Table 2. Production planning parameters in IFK**

| Resource Type/ Products Type | Bullion Type 1 (OC)/ (Tons) | Bullion Type 2 (FN)/ (Tons) | Bullion Type 3 (IR)/ (Tons) | Smelter Machines (Min) | Rolling Machines (Min) | Expected Profit (Million Rials* per Ton) |
|---|---|---|---|---|---|---|
| 8 mm | 0.5 | 0.1 | 0.4 | (25, 30, 33) | (55, 60, 64) | (95, 100, 110) |
| 10 mm | 0.2 | 0.2 | 0.65 | (30, 35, 39) | (60, 65, 69) | (100, 105, 110) |
| 12 mm | 0.8 | 0.3 | 0.15 | (35, 40, 45) | (90, 95, 100) | (90, 95, 100) |
| 14 mm | 0.9 | 0.5 | 0.05 | (42, 46, 52) | (96, 100, 105) | (100, 110, 115) |
| 16 mm | 0.2 | 0.5 | 0.45 | (50, 55, 58) | (100, 105, 109) | (98, 105, 112) |
| 18 mm | 0.1 | 0.8 | 0.35 | (55, 60, 63) | (105, 110, 115) | (95, 101, 106) |
| 20 mm | 0.1 | 0.7 | 0.09 | (55, 62, 65) | (130, 135, 140) | (110, 120, 128) |
| 22 mm | 0.3 | 0.6 | 0.2 | (60, 65, 69) | (140, 145, 150) | (140, 150, 158) |
| 24 mm | 0.4 | 0.5 | 0.5 | (80, 85, 89) | (145, 150, 159) | (155, 165, 171) |
| Capacity of Each Resource | 1200 (Per Month) | 900 (Per Month) | 1050 (Per Month) | 900 (Per Day**) | 2000 (Per Day) | |

* Iranian currency (IRR) ** 22 Working Days per Month

Considering each final product of fittings from 8 mm to 24 mm as decision variables, $x_1, x_2, \ldots, x_9$ , each bullion availability and smelter and rolling stages capacity as constraints, based on mentioned expected profit as coefficient of each decision variable in objective function, for maximizing IFK Company, a linear production program is modeled as below.

$$
\begin{aligned}
& Max Z = (95,100,110)x_1 + (100,105,110)x_2 + (90,95,100)x_3 + (100,110,115)x_4 + (98,105,112)x_5 \\
& + (95,100,105)x_6 + (110,120,130)x_7 + (140,150,160)x_8 + (155,165,170)x_9 \\
& \text{Subject To:} \\
& \text{Bullion 1) } 0.5x_1 + 0.2x_2 + 0.8x_3 + 0.9x_4 + 0.2x_5 + 0.1x_6 + 0.1x_7 + 0.3x_8 + 0.4x_9 \le 1200 \\
& \text{Bullion 2) } 0.1x_1 + 0.2x_2 + 0.3x_3 + 0.5x_4 + 0.5x_5 + 0.8x_6 + 0.7x_7 + 0.6x_8 + 0.5x_9 \le 900 \\
& \text{Bullion 3) } 0.4x_1 + 0.65x_2 + 0.15x_3 + 0.05x_4 + 0.45x_5 + 0.35x_6 + 0.09x_7 + 0.2x_8 + 0.5x_9 \le 1050 \\
& \text{Smelter) } (25,30,33)x_1 + (30,35,39)x_2 + (35,40,45)x_3 + (42,46,52)x_4 + (50,55,58)x_5 + (55,60,63)x_6 \\
& + (55,62,65)x_7 + (60,65,69)x_8 + (80,85,89)x_9 \le 22*900 \\
& \text{Rolling) } (55,60,64)x_1 + (60,65,69)x_2 + (90,95,100)x_3 + (96,100,105)x_4 + (100,105,109)x_5 \\
& + (105,110,115)x_6 + (130,135,140)x_7 + (140,145,150)x_8 + (145,150,159)x_9 \le 22*2000 \\
& x_j \ge 0,\ j = \{1,2,\ldots,9\}
\end{aligned}
$$

Having solved the above production planning problem, the results are obtained as shown in table 3.

**Table 3. Solutions for different levels of *α* in fuzzy production problem of IFK**

| α-level | Fuzzy objective value | α-level | Fuzzy objective value |
|---|---|---|---|
| 0 | (41524.62, 44153.85, 46983.08) | 0.6 | (43758.02, 46512.16, 49466.31) |
| 0.1 | (41906.84, 44555.18, 47403.52) | 0.7 | (44077.24, 46880.96, 49837.82) |
| 0.2 | (42297.07, 44964.92, 47832.77) | 0.8 | (44415.42, 47252.97, 50226.74) |
| 0.3 | (42695.56, 45383.33, 48271.11) | 0.9 | (44762.32, 47634.55, 50625.67) |
| 0.4 | (43102.57, 45810.7, 48718.82) | 1.0 | (45118.26, 48026.09, 51035) |
| 0.5 | (43432.54, 46163.43, 49094.33) | | |

Here in this case,

$$
f_M(x) = \begin{cases} (x - 41524.62)/9510.38, & 41524.62 \le x \le 51035 \\ 0, & \text{otherwise} \end{cases}
$$

By applying Eqs. (37) to (39), the best solution is chosen as (45118.26, 48026.09, 51035) with optimal variables being determined as follows: $x_1^* = x_2^* = 100, x_4^* = 49.78, x_3^* = x_5^* = x_6^* = x_7^* = x_8^* = x_9^* = 30$

## 6 Conclusion

Fuzzy linear programming is an interesting and widely studied field which different methods and procedures are proposed to solve these problems. In this paper a new approach is developed to solve the FLP problems based on the notion of minimization and maximization sets as a ranking function of the powerful fuzzy numbers. The main idea of the proposed method is to maximize or minimize the total utility of the objective function, as an aggregated function of its intersection with the minimization and maximization sets. These sets are subjected to a set of fuzzy constraints which are transformed into a set of parametric linear constraints later based on $\alpha$-cuts. Afterwards, the process of solving the FLP problem is repeated for different values of $\alpha$. Finally, the best solution is chosen by considering the intersection between the objective function values obtained for different levels of $\alpha$ with the minimization or maximization sets of the objective function. Performance of the suggested method studied in two numerical examples.

The results of the proposed method, in comparison with the two previously extended methods, shows that its result may provide a greater satisfaction for solving the FLP problems. The idea of this paper also can be simply extended to the case of full FLP problems, where both variables and parameters are defined as fuzzy numbers and variables, respectively.

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