Mathematical Modelling and Analysis
Volume 25, Issue 2, 167–183, 2020
https://doi.org/10.3846/mma.2020.8023

http://mma.vgtu.lt
ISSN: 1392-6292
eISSN: 1648-3510

M-matrices and Convergence of Finite Difference Scheme for Parabolic Equation with an Integral Boundary Condition

Regimantas Čiupaila[a], Mifodijus Sapagovas[b] and Kristina Pupalaigė[c]

[a] *Vilnius Gediminas Technical University*
Saulėtekio av. 11, LT-10223, Vilnius, Lithuania
[b] *Faculty of Mathematics and Informatics, Vilnius University*
Akademijos str. 4, LT-08412, Vilnius, Lithuania
[c] *Department of Applied Mathematics, Kaunas University of Technology*
Studentų str. 50, LT-51368, Kaunas, Lithuania
E-mail(*corresp.*): `kristina.pupalaige@ktu.lt`
E-mail: `regimantas.ciupaila@vgtu.lt`
E-mail: `mifodijus.sapagovas@mii.vu.lt`

Received January 28, 2019; revised December 14, 2019; accepted December 16, 2019

Abstract. In the paper, the stability and convergence of difference schemes approximating semilinear parabolic equation with a nonlocal condition are considered. The proof is based on the properties of M-matrices, not requiring the symmetry or diagonal predominance of difference problem. The main presumption is that all the eigenvalues of the corresponding difference problem with nonlocal conditions are positive.

Keywords: finite difference method, nonlocal boundary condition, convergence, M-matrices.

AMS Subject Classification: 65M06; 65M12; 65N25.

1 Introduction

Initiations in the research of parabolic equations with nonlocal conditions appeared in papers [3, 19] more than half a century ago. New problems, formulated in the papers mentioned, became as the object of investigations of many authors later on.

During the last decades many new papers have appeared, where solution of parabolic equations with integral boundary conditions by the finite difference method was considered in various aspects, including the stability and convergence of difference schemes. In paper [11], the estimation of error in the maximum norm was obtained for the one-dimensional linear parabolic equation with integral boundary conditions.The analogous result was provided in [12] for the numerical solution of a semilinear problem with integral boundary conditions. Various aspects of difference schemes, such as convergence, stability, algorithms of realization, approximation of increased accuracy in one-dimensional case were investigated in many articles (see, for example [8, 9, 22, 26] and the references therein). The influence of complex coefficients on the stability of difference schemes was investigated in [20, 31].

In a two-dimensional case of parabolic equations with integral conditions, the difference methods were applied in [4, 6, 21, 24, 29], and to the equations with other type of nonlocal conditions – in [13, 15].

We emphasize the fact that most of theoretical issues of difference schemes were investigated in line with the sufficiently strong conditions introduced (it is not clear whether they are necessary) for the various parameters or functions in nonlocal conditions (see, for example [11, 12, 21]), or proving the stability or convergence in unusual energetic norms (see [13, 16]).

Another important aspect is that theoretical investigations in the numerical analysis are strongly influenced by practical applications in heat conductions [3], thermoelasticity [10], underground water flow [23], electrochemistry [5], and so on. In [7, 14], some mathematical models with nonlocal conditions for bioreactors are described.

In this paper, the stability and convergence of difference schemes for two-dimensional nonlinear parabolic equations with an integral boundary condition are proved using the properties of M-matrices and the structure of spectrum of difference operators with nonlocal conditions.

In papers [28, 33, 38], the theory of M-matrices was used for the investigation of difference systems obtained from the elliptic equation with an integral condition. In many cases, the matrix of the system of difference equations, approximating a differential equation with nonlocal conditions, is characterized by the properties specific to M-matrices. These properties might be used to obtain the conditions of convergence of iterative methods [28, 38]. In the paper [18], the properties of M-matrices were applied to the investigation of the stability of difference schemes in the energetic norm.

In the paper [33], using the properties of M-matrices, the error of the solution of the difference problem for nonlinear elliptic equation with nonlocal boundary condition was estimated. Two aspects of such a method of proof have been noted. First, the main idea of error estimation is the construction of majorant, similar as applying the maximum principle method. Second, in the M-matrix method, the error estimate is proved declining the diagonal predominance property of the matrix for the system of difference equations, which is necessary for the maximum principle method. The results of the investigation of the structure of the difference operator spectrum are used instead of the diagonal predominance property.

The stucture of the spectrum of difference operators with nonlocal conditions was investigated in many papers (see [30, 35, 36] and also references in review article [37]). In the papers [2, 16, 17, 27, 34], the stability of difference schemes in the energetic norm according to the structure of the spectrum is examined.

In this article the methodology of the estimation of the error of the solution as in [33] is applied to a new class of problems with nonlocal conditions, namely, to nonlinear two-dimensional parabolic equations. The main result of the present paper is that the stability and convergence of difference schemes for a parabolic equation with an integral boundary condition in the uniform norm has been proved using the structure of the spectrum of difference operator and the theory of M-matrices. The statement on the convergence of the difference scheme in the uniform norm was proven only in the case of nonlocal conditions under which the matrix of the system of difference equations is diagonally dominant (see, for example [11, 12, 21]). Using the methodology of M-matices the requirement of diagonal dominance was abolished.

The paper is organized as follows. In Section 2, the differential and difference problems are stated. The difference problem for the error of the solution is investigated in Section 3. The latter problem is expressed in the matrix form as a two-layer difference scheme. In Section 4, some new formulations of the well-known properties of M-matrices, adapted to the system of difference equations are obtained. Using these properties, the error estimation is presented for the solution and the convergence of the difference scheme in the uniform norm is proved in Section 5 . In Section 6, a short analysis of the stability of a difference scheme is provided. Results of numerical experiments are delivered in Section 7. Some final conclusions are presented in Section 8.

2 Statement of differential and difference problem

A semilinear two-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f(x, y, t, u) + p(x, y, t) \tag{2.1}$$

is considered in the rectangular domain $D = \{0 < x < 1,\ 0 < y < 1\}$ and $t \in (0, T]$. The initial condition

$$u(x, y, 0) = \varphi(x, y), \qquad (x, y) \in D \tag{2.2}$$

and boundary conditions

$$u(0, y, t) = \gamma \int_0^1 u(x, y, t)dx + \mu_1(y, t), \tag{2.3}$$

$$u(1, y, t) = \mu_2(y, t), \quad u(x, 0, t) = \mu_3(x, t), \quad u(x, 1, t) = \mu_4(x, t), \tag{2.4}$$

where f, p, φ, μ_i, $i = 1, 2, 3, 4$ are given sufficiently smooth functions, are formulated. Furthermore, the functions μ_2, μ_3 and μ_4 satisfy compatibility conditions at the points $(1, 0)$ and $(1, 1)$. In additional, boundary conditions (2.3) and (2.4) must be satisfied for initial function $\varphi(x, y)$.

The following hypotheses are assumed to be true:

- Hypothesis H1. $\frac{\partial f}{\partial u} \geq 0$ for all the values $(x, y) \in D$ and $u(x, y, t)$;
- Hypothesis H2. γ is a given real number $0 \leq \gamma < 2 - \rho$, $\rho \in (0, 2]$.

A difference problem approximating differential problem (2.1)–(2.4) is written. To this end the followin notation is introduced: $h = \frac{1}{N}$, $\tau = \frac{T}{N_1}$, where N and N_1 are positive integer numbers. Further, let $x_i = ih$, $y_j = jh$. $i, j = 0, 1, \ldots, N$; $t^n = n\tau$, $n = 0, 1, \ldots, N_1$. A numerical approximation U_{ij}^n to the exact solution $u_{ij}^n = u(x_i, y_j, t^n)$ of differential problem (2.1)–(2.4) is considered.

The following notation is presented

$$\partial_x^2 U_{ij}^n = \frac{U_{i-1,j}^n - 2U_{ij}^n + U_{i+1,j}^n}{h^2}, \quad \partial_y^2 U_{ij}^n = \frac{U_{i,j-1}^n - 2U_{ij}^n + U_{i,j+1}^n}{h^2},$$
$$\partial_t U_{ij}^n = \frac{U_{ij}^n - U_{ij}^{n-1}}{\tau}, \quad l_j(U_{ij}^n) = h\left(\frac{U_{0j}^n + U_{Nj}^n}{2} + \sum_{i=1}^{N-1} U_{ij}^n\right).$$

A finite difference scheme is now defined

$$\partial_t U_{ij}^n = \partial_x^2 U_{ij}^n + \partial_y^2 U_{ij}^n - f(U_{ij}^n) + p_{ij}^n, \ i, j = 1, 2, \ldots, N-1, \ n = 1, \ldots, N_1, \quad (2.5)$$
$$U_{0j}^n = \gamma l_j(U_{ij}^n) + (\mu_1)_j^n, \quad j = 1, 2, \ldots, N-1; \quad (2.6)$$
$$U_{ij}^0 = \varphi_{ij}, \quad i, j = 0, 1, \ldots, N, \quad (2.7)$$
$$U_{Nj}^n = (\mu_2)_j^2, \ U_{i0}^n = (\mu_3)_i^n, \ U_{iN}^n = (\mu_4)_i^n, \quad i, j = 1, 2, \ldots, N-1. \quad (2.8)$$

Further in this article for shorter writing no restrictions are imposed on index n in difference equations or conditions, assuming that n is any integer $1 \leq n \leq N_1$.

The solution $U_{i,j}^{n-1}$ at the all points (i, j) of the layer $t = t^{n-1}$ is known. Then system (2.5)–(2.8) for the unknowns U_{ij}^n with fixed n can be interpreted formally as difference analogue of some elliptic equation. The solution of such system by the iterative methods was investigated in [28]. It follows from these results that under hypotheses H1 and H2, the unique solution U_{ij}^n of the system of difference equations (2.5)–(2.8) exists.

Thus the following statement is obtained.

Lemma 1. *If the hypotheses H1 and H2 are true, then unique solution of the system of difference equations* (2.5)–(2.8) *exists.*

Let the differential problem (2.1)–(2.4) possess the unique smooth enough solution, i.e. the derivatives of the solution with respect to x and y up to the fourth order and the first and second derivatives with respect to t are bounded. Then the error of approximation is $O(h^2 + \tau)$. So, the following difference problem for the solution u_{ij}^n of the differential problem can be written:

$$\partial_t u_{ij}^n = \partial_x^2 u_{ij}^n + \partial_y^2 u_{ij}^n - f(u_{ij}^n) + p_{ij}^n + R_{ij}^n + r_{ij}^n, \ i, j = 1, 2, \ldots, N-1, \quad (2.9)$$
$$u_{0j}^n = \gamma l_j(u_{ij}^n) + (\mu_1)_j^n + R_j^n, \quad j = 1, 2, \ldots, N-1, \quad (2.10)$$
$$u_{ij}^0 = \varphi_{ij}, \quad i, j = 0, 1, \ldots, N, \quad (2.11)$$
$$u_{Nj}^n = (\mu_2)_j^2, \ u_{i0}^n = (\mu_3)_i^n, \ u_{iN}^n = (\mu_4)_i^n, \ i, j = 1, 2, \ldots, N-1. \quad (2.12)$$

Here, according to the presumption on the smoothness of solution of the differential problem, the following estimations are true:

$$\left|R_{ij}^n\right| \le \frac{h^2}{6} M_4, \quad \left|r_{ij}^n\right| \le \frac{\tau}{2} C_2, \quad \left|R_j^n\right| \le \frac{h^2}{12} M_2, \tag{2.13}$$

where

$$M_4 = \max\left(\left|\frac{\partial^4 u}{\partial x^4}\right|, \left|\frac{\partial^4 u}{\partial y^4}\right|\right), \quad M_2 = \max\left|\frac{\partial^2 u}{\partial x^2}\right|, \quad C_2 = \max\left|\frac{\partial^2 u}{\partial t^2}\right|.$$

3 Difference problem for the error

The error of the solution is noted $z_{ij}^n = u_{ij}^n - U_{ij}^n$. From the problems (2.5)–(2.8) and (2.9)–(2.12) the following system recieved:

$$\partial_t z_{ij}^n = \partial_x^2 z_{ij}^n + \partial_y^2 z_{ij}^n - d_{ij}^n z_{ij}^n + R_{ij}^n + r_{ij}^n, \quad i,j = 1,2,\dots,N-1, \tag{3.1}$$

$$z_{0j}^n = \gamma l_j(z_{ij}^n) + R_j^n, \quad j = 1,2,\dots,N-1, \tag{3.2}$$

$$z_{ij}^0 = 0, \quad i,j = 0,1,\dots,N, \tag{3.3}$$

$$z_{Nj}^n = z_{i0}^n = z_{iN}^n = 0, \quad i,j = 1,2,\dots,N-1, \tag{3.4}$$

where the following is denoted

$$d_{ij}^n = \frac{\partial f(\tilde{u}_{ij}^n)}{\partial \tilde{u}_{ij}^n}, \tag{3.5}$$

and $\tilde{u}_{ij}^n$ is a certain intermediate value $\tilde{u}_{ij}^n = u_{ij}^n + \theta U_{ij}^n$, $|\theta| \le 1$.

To estimate the error z_{ij}^n, first of all this system is rearranged. In this system, where n is a fixed number, there are $N(N-1)$ equations and the same number of unknowns. This system is reduced to two separate systems of lower order.

With this aim z_{0j}^n is expressed from equation (3.2) for each $j = 1,2,\dots,N-1$:

$$z_{0j}^n = \alpha \sum_{i=1}^{N-1} z_{ij}^n + \beta R_j^n, \quad j = 1,2,\dots,N-1, \tag{3.6}$$

where $\alpha = \gamma h/(1-\frac{\gamma h}{2})$, $\beta = 1/(1-\frac{\gamma h}{2})$. As $h \le \frac{1}{2}$ and according to hypothesis H2, $0 \le \gamma < 2$, then

$$0 \le \alpha \le 2\gamma h < 2, \quad 1 \le \beta \le 2. \tag{3.7}$$

Putting expression (3.6) into equations (3.1), as $i = 1$ and introducing the new notation, for each fixed $n \ge 1$ is obtained:

$$\begin{aligned} L(z_{1j}^n) :=& \partial_t z_{1j}^n - \frac{1}{h^2}\left(\alpha \sum_{i=1}^{N-1} z_{ij}^n - 2z_{1j}^n + z_{2j}^n\right) - \partial_y^n z_{1j}^n + d_{1j}^n z_{1j}^n \\ =& R_{1j}^n + r_{1j}^n + \beta R_j^n/h^2, \quad j = 1,2,\dots,N-1, \end{aligned} \tag{3.8}$$

$$L(z_{ij}^n) := \partial_t z_{ij}^n - \partial_x^2 z_{ij}^n - \partial_y^2 z_{ij}^n + d_{ij}^n z_{ij}^n = R_{ij}^n + r_{ij}^n, \quad {\scriptstyle i=2,3,\dots,N-1, \atop j=1,2,\dots,N-1.} \tag{3.9}$$

For each fixed $n \geq 1$ equations (3.1), (3.2) and (3.6), (3.8), (3.9) are equivalent. But now the former system (3.1), (3.2) and (3.4) is reduced to the separate systems:

– $(N-1)^2$ order system (3.8), (3.9), (3.4) with the unknowns z_{ij}^n in the interior points of the grid area, i.e. as $i, j = 1, 2, \dots, N-1$ and

– $N-1$ order system (3.6) with the unknowns z_{0j}^n, $i, j = 1, 2, \dots, N-1$; indeed, this system is the explicit formulas providing the opportunity to calculate z_{0j}^n, when the solution of the first system z_{ij}^n, $i, j = 1, 2, \dots, N-1$ is known.

Now the system (3.1), (3.2), (3.4) for fixed $n \geq 1$ is written in the matrix form

$$Az^n = Bz^{n-1} + r^n, \tag{3.10}$$

where A, B are matrices of order $(N-1)^2$, z^n and r^n are vectors of the same order. To this end, as usual, the following order of components of vector $z^n = \{z_{ij}^n\}$, $i, j = 1, 2, \dots, N-1$ is used: first the auxiliary vectors of order $N-1$ are defined, $z_j^n = \left(z_{1j}^n,\ z_{2j}^n, \dots, z_{N-1,j}^n\right)^T$, $j = 1, 2, \dots, N-1$, and afterwards $z^n = \left(z_1^n, z_2^n, \dots, z_{N-1}^n\right)^T$.

Now the system (3.1), (3.2), (3.4) can be written in the following matrix form

$$\left(\tau^{-1}I + \Lambda - C + D\right) z^n = \tau^{-1}Iz^{n-1} + r^n.$$

Matrices I, Λ, C and D of order $(N-1)^2$ included to this system are formed as follow. I is the unique matrix. Λ is the matrix, corresponding to the difference Laplace operator (with a minus sign) with zero Dirichlet conditions. C is the matrix composed of the coefficients in the member $h^{-2}\alpha \sum\limits_{i=1}^{N-1} z_{ij}^n$ of equation (3.8). D is the diagonal matrix with the elements d_{ij}^n, prescribed by formula (3.5). For details see [33]. Denote

$$A_1 = \tau^{-1}I + \Lambda - C, \quad A = A_1 + D, \quad B = \tau^{-1}I. \tag{3.11}$$

Thus, the system of difference equations is written in the form of (3.10). Another form of this system is

$$(A_1 + D)z^n = Bz^{n-1} + r^n. \tag{3.12}$$

The solution of system (3.10) is estimated using the properties of M-matrices.

4 M-matrices and the system of difference equations

In this section, several properties of M-matrices are formulated and used them for the investigation of system (3.10).

Definition 1. [1]. The square matrix $A = \{a_{kl}\}$, $k, l = \overline{1, m}$ is called an M-matrix if $a_{kl} \leq 0$ as $k \neq l$ and if there exists the inverse matrix A^{-1}, all the elements of which are nonnegative ($A^{-1} \geq 0$).

In the present paper, the following notation is used: $A > 0$ ($A \geq 0$) if $a_{kl} > 0$ ($a_{kl} \geq 0$) for all k, l. Also $A < B$ ($A \leq B$), if $a_{kl} < b_{kl}$ ($a_{kl} \leq b_{kl}$) is written. Analogous notation will also be used for the vectors.

The following three lemmas could be interpreted as the new statement of certain properties of M-matrices, adapted to the systems of difference equations approximating parabolic equations. As far as the authors are acquainted, these properties of M-matrices were not formulated in this form so far.

Lemma 2. *Let v^n be the solution of the difference equation*

$$Av^n = Bv^{n-1} + f^n, \quad n \geq 1. \tag{4.1}$$

If A is an M-matrix, $B \geq 0$, $v^0 \geq 0$ and $f^n \geq 0$ for all $n \geq 1$, then $v^n \geq 0$ for all $n \geq 1$.

Proof. The statement of Lemma 2 follows by using the method of mathematical induction. We have $v^0 \geq 0$. Let $v^{n-1} \geq 0$, then

$$v^n = A^{-1}Bv^{n-1} + A^{-1}f^n \geq 0.$$

□

Suppose there is the vector v with the coordinates v_k, $k = 1, 2, \ldots, m$. Vector with the coordinates $|v_k|$ is designated by $|v|$, i.e. $|v| = \{|v_k|\}$.

Lemma 3. *Let v^n and w^n be solutions of difference equation* (4.1)*, and the equation*

$$Aw^n = Bw^{n-1} + g^n, \quad n \geq 1 \tag{4.2}$$

respectively. If A is an M-matrix, $B \geq 0$, $w^0 \geq 0$, $g^n \geq 0$ as $n \geq 1$ and, additionally, $|v^0| \leq w^0$, $|f^n| \leq g^n$, $n \geq 1$, then $|v^n| \leq w^n$, $n \geq 1$.

Proof. Just like in the proof of Lemma 2, where $w^{n-1} \geq 0$, assume that $|v^{n-1}| \leq w^{n-1}$. Then

$$\begin{aligned} |v^n| =& |A^{-1}Bv^{n-1} + A^{-1}f^n| \leq A^{-1}B|v^{n-1}| + A^{-1}|f^n| \\ \leq& A^{-1}Bw^{n-1} + A^{-1}g^n = w^n \end{aligned}$$

is derived from (4.1) and (4.2). □

Lemma 4. *Let v^n and w^n be solutions of difference equations*

$$\begin{aligned} (A + D)v^n =& Bv^{n-1} + f^n, \quad n \geq 1, \\ Aw^n =& Bw^{n-1} + g^n, \quad n \geq 1, \end{aligned}$$

respectively, where $D = \{d_{kk}\}$ is a diagonal matrix $D \geq 0$. If A is an M-matrix, $B \geq 0$, $w^0 \geq 0$, $g^n \geq 0$, as $n \geq 1$ and $|v^0| \leq w^0$, $|f^n| \leq g^n$, $n \geq 1$ additionally, then $|v^n| \leq w^n$, $n \geq 1$.

Proof. When $D \geq 0$ is a diagonal matrix and A is an M-matrix, then $A+D$ is also an M-matrix. Moreover

$$A^{-1} \geq (A+D)^{-1} \geq 0.$$

Similary as in the proofs of Lemmas 2 and 3, from $|v^{n-1}| \leq w^{n-1}$ it follows

$$\begin{aligned} |v^n| =& |(A+D)^{-1}Bv^{n-1} + (A+D)^{-1}f^n| \\ \leq& (A+D)^{-1}B|v^{n-1}| + (A+D)^{-1}|f^n| \leq A^{-1}Bw^{n-1} + A^{-1}g^n = w^n. \end{aligned}$$

□

5 Error estimation and convergence of the difference scheme

With reference to the results of Section 4 , **we estimate** the solution of the system of difference equations (3.1)–(3.4), i.e. the error $z_{ij}^n = u_{ij}^n - U_{ij}^n$. In Section 3, for each fixed $n \geq 1$ this system was reduced to two systems: system (3.2), (3.9), (3.4) and explicit formulas (3.6). The first system was presented in matrix form (3.10), where matrices A and B were described by formulas (3.11), and the vector $\tilde{r}^n$ was composed of the expressions of the right-hand sides of equations (3.4), (3.9), i.e. $\tilde{r}^n = \{\tilde{r}_{ij}^n\}$:

$$\tilde{r}_{ij}^n = \begin{cases} R_{1j}^n + r_{1j}^n + \dfrac{\beta R_j^n}{h^2}, & i = 1, \\ R_{ij}^n + r_{ij}^n, & i = 2, 3, \ldots, N-1. \end{cases}$$

Lemma 5. *If the hypotheses H1 and H2 are true, then matrix A of the system of difference equations* (3.10) *is an M-matrix.*

Proof. In the article [33] is proved, that the matrix $\Lambda - C$ under conditions H1 and H2 are M-matrices (see [33] Corollary 1). Since $I \geq 0$ and $D \geq 0$ are diagonal matrices, it follows from the properties of M-matrices that matrices $A_1 = \tau^{-1}I + \Lambda - C$ and $A = \tau^{-1}I + \Lambda - C + D$ are M-matrices as well. □

Theorem 1. *If the hypotheses H1 and H2 are true and estimations* (2.13) *for the error of approximation of differential problem* (2.1)–(2.4) *are valid, then for the solution of the system of difference equations* (3.1)–(3.4) *the following estimation is true:*

$$|z_{ij}^n| \leq C_3 h^2 + C_4 \tau, \quad i = 0, 1, \ldots, N-1; \quad j = 1, 2, \ldots, N-1,$$

where constants C_3 and C_4 do not depend on h and τ.

Proof. First of all, it is estimated that the solution z_{ij}^n of this system, as $i, j = 1, 2, \ldots, N-1$. In other words, in the beginning, the solution of system (3.8), (3.9), (3.4) is estimated. Two separate cases: $0 \leq \gamma \leq 1$ and $1 \leq \gamma \leq 2-\rho$ are considered.

An auxiliary function (majorant) is defined as:

$$w(x, y, t) = w_1(x, t) + w_2(x, y), \tag{5.1}$$

Case 1: $0 \leq \gamma \leq 1$. In this case,

$$w_1(x, t) = 0.5C_2(t + 1)\tau(2 - x)2, \tag{5.2}$$

$$w_2(x, y) = \frac{Mh^2}{24K}\left(1 - Kx^2 - Ky^2 - (1 - 2K)x\right). \tag{5.3}$$

Here $K = \rho/13$, C_2 is a constant described by inequalities (2.13), $M = \max(M_2, M_4)$. We admit, that function $w_2(x, y)$ was defined in [33] as the majorant of the problem in the case of elliptic equations. But in the case of parabolic equation the majorant must depend on the variable t. For the function $w(x, y, t)$ a system, analogous to system (3.1)–(3.4) choosing $d_{ij}^n = 0$ is written:

$$\partial_t w_{ij}^n = \partial_x^2 w_{ij}^n + \partial_y^2 w_{ij}^n + g_{ij}^n, \quad i, j = 1, 2, \ldots, N - 1, \tag{5.4}$$

$$w_{0j}^n = \gamma l_j(w_{ij}^n) + g_{0j}^n, \quad j = 1, 2, \ldots, N - 1, \tag{5.5}$$

$$w_{ij}^0 = g_{ij}^0, \quad i, j = 0, 1, \ldots, N, \tag{5.6}$$

$$w_{Nj}^n = g_{Nj}^n, \quad w_{i0}^n = g_{i0}^n, \quad w_{iN}^n = g_{iN}^n. \tag{5.7}$$

In this system g_{ij}^n, $i, j = 1, 2, \ldots, N-1$ and g_{0j}^n, $j = 1, 2, \ldots, N-1$ are unknown values as yet. They are calculated or evaluated using expressions (5.2), (5.3). By means of (5.1) the following are obtained

$$\frac{\partial w}{\partial t} = \frac{C_2\tau(2 - x)}{2}, \quad \frac{\partial^2 w}{\partial t^2} = 0,$$
$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial y^2} = -\frac{Mh^2}{12}, \quad \frac{\partial^4 w}{\partial x^4} = \frac{\partial^4 w}{\partial y^4} = 0.$$

Therefore,

$$\partial_t w_{ij}^n = \left(\frac{\partial w}{\partial t}\right)_{ij}^n, \quad \partial_x^2 w_{ij}^n + \partial_y^2 w_{ij}^n = \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)_{ij}^n.$$

Thus,

$$g_{ij}^n = \frac{C_2\tau(2 - x)}{2} + \frac{Mh^2}{6} \geq \frac{C_2\tau}{2} + \frac{Mh^2}{6}, \quad i, j = 1, 2, \ldots, N - 1. \tag{5.8}$$

As it follows from (5.2), (5.3) that

$$w(x, y, t) \geq w(1, 1, t) \geq 0.5C_0(t + 1)\tau \geq 0.5C_0\tau,$$

so

$$g_{ij}^0 \geq 0, \quad g_{Nj}^n \geq 0, \quad g_{i0}^n \geq 0, \quad g_{iN}^n \geq 0. \tag{5.9}$$

Let us estimate g_{0j}^n. Since $\frac{\partial^2 w}{\partial x^2} < 0$, then from the trapezoid formula the following is achieved [33]

$$l_j(w_{ij}^n) = \int_0^1 w(x, y, t)dx + \frac{h^2}{12}\frac{\partial^2 w}{\partial x^2} < \int_0^1 w(x, y, t)dx.$$

Thus $g_{0j}^n = (g_1)_{0j}^n + (g_2)_{0j}^n$, where

$$\begin{aligned}(g_1)_{0j}^n =&(w_1)_{0j}^n - \gamma l_j\left((w_1)_{0j}^n\right) > w_1(0,t^n) - \gamma w_1(x,t^n)dx\\ =&C_2(t^n+1)\tau - \frac{3}{4}\gamma C_2(t^n+1)\tau \geq \frac{C_2(t^n+1)\tau}{4} > 0; \qquad (5.10)\\ (g_2)_{0j}^n =&w_2(0,y_j) - \gamma l_j\left((w_2)_{0j}\right) > Mh^2/6.\end{aligned}$$

The last estimation have been proved in [33]. Thus,

$$g_{0j}^n > Mh^2/6. \qquad (5.11)$$

Case 2: $1 \leq \gamma < 2-\rho$. In this case in the definition of the function $w(x,y,t)$ only $w_1(x,t)$ is modified:

$$w_1(x,t) = \frac{C(t+1)\tau}{2}\left(\frac{\gamma}{2-\gamma} - \frac{2\gamma-2}{2-\gamma}x\right).$$

In this case, $(g_1)_{0j}^n = 0$ is obtained instead (5.10). The other three estimations (5.8), (5.9), and (5.11) remain the same.

In both cases, for the function $w(x,y,t)$ system (5.4)–(5.7) with the estimations of right-hand sides (5.8), (5.9), and (5.11) is received.

Now system (5.4)–(5.7) for each fixed $n \geq 1$, analogously to system (3.1)–(3.4), is reduced to two separate systems (see (3.6), and (3.8), (3.9)):

$$w_{0j}^n = \alpha\sum_{i=1}^{N-1} w_{ij}^n + \beta g_{0j}^n, \quad j=1,2,\dots,N-1,$$

where α and β satisfy the conditions (3.7) and

$$\begin{aligned}L(w_{1j}^n) =&g_{1j}^n + \frac{\beta g_{0j}^n}{h^2}, \quad j=1,2,\dots,N-1, \qquad (5.12)\\ L(w_{ij}^n) =&g_{ij}^n, \quad i=2,\dots,N-1;\ j=1,2,\dots,N-1. \qquad (5.13)\end{aligned}$$

The system (5.12)–(5.13) with boundary conditions (5.7) in matrix form (3.10) is written as it has been done earlier for system (3.8)–(3.9):

$$Aw^n = Bw^{n-1} + \tilde{g}^n, \qquad (5.14)$$

A and B are the same matrices as in (3.11). Since w_{ij}^n, differently from z_{ij}^n, is not equal to zero, as $n=0$ and i or j are equal to 1 or $N-1$, the expression $\tilde{g}_{ij}^n$ may not be coincident with g_{ij}^n, but satisfies the condition

$$\tilde{g}_{ij}^n = g_{ij}^n + \frac{\theta w_{ij}^n}{h^2}, \text{ where } \theta = \begin{cases}1, & \text{as } i=1,N-1 \text{ or } j=1,N-1,\\ 0, & \text{in other cases.}\end{cases}$$

So

$$\tilde{g}_{ij}^n \geq g_{ij}^n, \quad i,j=0,1,\dots,N-1.$$

Taking into account formulas (5.8), (5.9) and (5.11), it is that

$$\tilde{g}^n_{ij} = \begin{cases} \tilde{g}^n_{1j} \geq \dfrac{C_2\tau}{2} + \dfrac{Mh^2}{6} + \dfrac{\beta}{h^2}\dfrac{Mh^2}{6}, \\ \tilde{g}^n_{ij} \geq \dfrac{C_2\tau}{2} + \dfrac{Mh^2}{6}, \quad i = 2, 3, \ldots, N-1. \end{cases} \tag{5.15}$$

According to Lemma 5, matrix A of the system of equations (3.10), (5.14) is an M-matrix. From (3.12) and (5.15), with respect to (2.13), it follows that $|\tilde{r}^n| \leq \tilde{g}^n$. For initial and boundary conditions (3.3), (3.4), and (5.9) the inequalities

$$|z^0_{ij}| \leq w^0_{ij}, \quad |z^n_{iN}| \leq w^n_{iN}, \quad |z^n_{0j}| \leq w^n_{0j}, \quad |z^n_{Nj}| \leq w^n_{Nj}$$

are truthful as well.

Thus, for the systems of equations (3.10) and (5.14) all the presumptions of Lemmas 3 and 4 are correct. Hence it is derived

$$|z^n_{ij}| \leq w^n_{ij}, \quad i, j = 1, 2, \ldots, N-1, \; n = 1, \ldots, N_1.$$

In such way, from the expression of the function $w(x, y, t)$ the following recieved

$$|z^n_{ij}| \leq \frac{C_2(T+1)\tau}{2} + \frac{13Mh^2}{24\rho}, \quad i, j = 1, 2, \ldots, N-1, \; n = 1, \ldots, N_1. \tag{5.16}$$

Now, according to formula (3.6), it is found

$$|z^n_{0j}| \leq 2\gamma h \sum_{i=1}^{N-1} |z^n_{ij}| + 2|R^n_j| \leq \frac{4C_2(T+1)\tau}{2} + \frac{52Mh^2}{24\rho} + \frac{2Mh^2}{6} \leq C_3h^2 + C_4\tau, \tag{5.17}$$

where $C_3 = 17Mh^2/(6\rho)$, $C_4 = 2C_2(T+1)$. □

From the estimations of error (5.16) and (5.17) it follows the convergence of difference scheme (2.5)–(2.8) in the uniform norm $||z|| = \max_{(i,j,n)} |z^n_{ij}|$.

6 The stability of a difference scheme

Estimation (5.17) of error could also be interpreted as statement on stability of the difference scheme. Explaining more in detail, the definition of stability is introduced. Since difference scheme (2.5)–(2.8) is nonlinear, the common concept of stability can be used. Namely, two problems are considered. The first problem is (2.5)–(2.8) with the data p^n_{ij}, $(\mu_k)^n_j$, $k = 1, 2$, $(\mu_l)^n_i$, $l = 3, 4$ and φ_{ij}. The solution of this problem is U^n_{ij}. The second problem is taken with the modified data $\tilde{p}^n_{ij}$, $(\tilde{\mu}_k)^n_j$, $(\tilde{\mu}_l)^n_i$ and $\tilde{\varphi}_{ij}$. The solution of this problem is denoted by $\tilde{U}^n_{ij}$.

Definition 2. Difference scheme (2.5)–(2.8) is stable if for every real $\varepsilon > 0$ the value $\delta = \delta(\varepsilon)$, not depending on h and τ, exists such that

$$|U^n_{ij} - \tilde{U}^n_{ij}| \leq \varepsilon, \quad i, j = 0, 1, \ldots, N; \; n = 1, 2, \ldots, N_1$$

if the data of both problems considered differs by not more than δ.

Denote

$$\tilde{z}_{ij}^n = U_{ij}^n - \tilde{U}_{ij}^n, \quad \delta_{ij}^0 = \varphi_{ij} - \tilde{\varphi}_{ij},$$
$$\delta_{kj}^n = (\mu_k)_j^n - (\tilde{\mu}_k)_j^n, \quad k = 1,2, \quad \delta_{il}^n = (\mu_l)_i^n - (\tilde{\mu}_l)_i^n, \quad l = 3,4.$$

For the unknowns $\tilde{z}_{ij}^n$, the following system of difference equations (see (3.1)–(3.4)) is written:

$$\partial_t \tilde{z}_{ij}^n = \partial_x^2 \tilde{z}_{ij}^n + \partial_y^2 \tilde{z}_{ij}^n - d_{ij}^n \tilde{z}_{ij}^n + \delta_{ij}^n, \quad i,j = 1,2,\dots,N-1, \tag{6.1}$$
$$\tilde{z}_{0j}^n = \gamma l_j(\tilde{z}_{ij}^n) + \delta_{0j}^n, \quad j = 1,2,\dots,N-1, \tag{6.2}$$
$$\tilde{z}_{ij}^0 = \delta_{ij}^0, \quad i,j = 0,1,\dots,N, \tag{6.3}$$
$$\tilde{z}_{Nj}^n = \delta_{Nj}^n, \quad \tilde{z}_{i0}^n = \delta_{i0}^n, \quad \tilde{z}_{iN}^n = \delta_{iN}^n, \quad i,j = 1,2,\dots,N-1, \tag{6.4}$$

for which the the following estimation is valid:

$$|\delta_{ij}^n| \le \delta, \quad i,j = 0,1,\dots,N; \; n \ge 0. \tag{6.5}$$

With this system we perform the same transformations as in Section 3 for system (3.1)–(3.4). The obtained analogue of system (3.2) is written

$$A\tilde{z}^n = B\tilde{z}^n + (\tilde{r}_1)^n,$$

where

$$(\tilde{r}_1)_{ij}^n = \begin{cases} \delta_{1j}^n + \beta\delta_{0j}^n/h^2, \\ \delta_{ij}^n, \quad i = 2,\dots,N-1. \end{cases}$$

Now the analogue of Theorem 1 can be formulated.

Theorem 2. *If the hypotheses H1 and H2 are true, then the system of difference equations* (2.5)–(2.8) *is stable in the sense of Definition 2.*

Proof. According to Definition 2, it is necessary to prove that for each $\varepsilon > 0$ the value $\delta = \delta(\varepsilon)$, not depending on h and τ, will arise, such that, for the solution of system of difference equations (6.1)–(6.4), the estimate

$$|\tilde{z}_{ij}^n| \le \varepsilon \tag{6.6}$$

is correct if (6.5) is valid. This statement could be fully proved analogously as Theorem 1. It is only enough to change the definition of the function $w(x,y,t)$ a little. Namely in the definition of the function $w_1(x,t)$ (formula (5.2)) the constant δ can be used instead of $\frac{C_2\tau}{2}$. In the definition of the function $w_2(x,y)$ (formula (5.3)) can be used δ instead of constant $\frac{Mh^2}{24}$. Then analogously to estimates (5.16) and (5.17), it is obtained

$$|\tilde{z}_{ij}^n| \le (T+1)\delta + 13\delta/\rho = (T+1+13/\rho)\,\delta, \quad i,j = 1,2,\dots,N-1 \tag{6.7}$$

and

$$|\tilde{z}_{0j}^n| \le (8T+12+13/\rho)\,\delta, \quad j = 1,2,\dots,N-1. \tag{6.8}$$

Thus if

$$\delta \le C_5\varepsilon, \quad C_5 = \min\{T+1+13/\rho,\; 8T+12+13/\rho\},$$

then (6.6) is satisfied. □

7 Numerical results

In order to verify the theoretical results and demonstrate the order of convergence of the approximate solution, some test problem where the exact solution of differential problem is known is considered.

A differential equation is considered

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u^3 + p(x,y,t)$$

in the domain $D = \{0 < x < 1,\ 0 < y < 1,\ 0 < t \le T\}$ with integral condition (2.3), boundary conditions (2.4) and initial condition (2.2). The functions $p(x,y,t)$, $\varphi(x,y)$, $\mu_i(y,t)$, $i = 1,2$, and $\nu_i(x,t)$, $i = 3.4$ are chosen that the function

$$u(x,y,t) = e^{at} \sin \pi x \sin \pi y + bx^2 \tag{7.1}$$

is the analytical solution of the problem under investigation. The test problem has been solved with some different parameters γ, a, b, T, h and τ.

In order to evaluate the accuracy of the numerical method, the absolute or relative error were used

$$E^n_{h,\tau} = \max_{i,j} \left| u(x_i, y_j, t^n) - U^n_{i,j} \right|, \quad \varepsilon^n_{h,\tau} = \max_{i,j} \frac{\left| u(x_i, y_j, t^n) - U^n_{i,j} \right|}{\left| u(x_i, y_j, t^n) \right|}.$$

The theoretical results presented in the previous section do not depend upon the numerical method that is used to solve system of nonlinear difference equations (2.5)–(2.8) with fixed n.

This system can be solved using one of the iterative methods suitable for problems with nonlocal conditions [28]. Here a Jacobi method was used.

The results of the numerical test for different h and τ are presented in Table 1. It reveals that the absolute error $E^n_{h,\tau}$ decreases approximately as $O(h^2 + \tau)$. Now the role of additional term bx^2 in (7.1) is explained. The

Table 1. The values of absolute error $E^n_{h,\tau}$ for different h and τ; $b = 1$; $T = 1$.

	$h = \frac{1}{10}$ $\tau = \frac{1}{40}$	$h = \frac{1}{20}$ $\tau = \frac{1}{160}$	$h = \frac{1}{40}$ $\tau = \frac{1}{640}$	$h = \frac{1}{80}$ $\tau = \frac{1}{2560}$
$a = 0.5,\ \gamma = 2$	0.0098	0.0025	$6.1565 \cdot 10^{-4}$	$1.5459 \cdot 10^{-4}$
$a = 0.5,\ \gamma = 1$	0.0098	0.0025	$6.1500 \cdot 10^{-4}$	$1.5347 \cdot 10^{-4}$
$a = 0.5,\ \gamma = 0.5$	0.0098	0.0025	$6.1420 \cdot 10^{-4}$	$1.5329 \cdot 10^{-4}$
$a = 0.5,\ \gamma = 0$	0.0098	0.0025	$6.1376 \cdot 10^{-4}$	$1.5319 \cdot 10^{-4}$
$a = 1,\ \gamma = 1$	0.0112	0.0028	$7.0765 \cdot 10^{-4}$	$1.9918 \cdot 10^{-4}$

constant M_2 in (2.13) depends on b. But constant M_4 in that formula does not depend on b. Thus, it is feasible, varying $b > 0$, investigate the influence of the approximation error of nonlocal condition (which is bounded by constant M_2) without changing approximation error of differential equation (which is bounded by constant M_4). The results are presented in Table 2.

Table 2. The values of absolute error $E^n_{h,\tau}$ for diferent b; $h = \frac{1}{40}$, $\tau = \frac{1}{640}$; $T = 1$.

	$b = 0$	$b = 5$	$b = 10$	$b = 15$
$a = 0.5,\ \gamma = 1$	$6.5273 \cdot 10^{-4}$	$5.4711 \cdot 10^{-4}$	0.0011	0.0016
$a = 0.5,\ \gamma = 2$	$6.1970 \cdot 10^{-4}$	0.0015	0.0030	0.0044
$a = 1,\ \gamma = 2$	$9.1335 \cdot 10^{-4}$	0.0015	0.0029	0.0044
$a = 2,\ \gamma = 2$	0.0044	0.0034	0.0029	0.0043

In Table 3, the results of numerical experiment of influence of T on the error of solution is presented. All constants M_2, M_4 and C_2 from (2.13) depend on e^{at} for solution (7.1). It means that in the case $a > 0$ these constants as well as the absolute error $E^n_{h,\tau}$ for fixed h and τ can grow rapidly, when t is growing. In case $a > 0$ the relative error $\varepsilon^n_{h,\tau}$ is used. The relative error with fixed h and τ increase slowly with the growing t (see the estimate of constant C_4 in (5.17), which depends on T). The numerical results confirm the stability of the method.

Table 3. The values of errors $E^n_{h,\tau}$ and $\varepsilon^n_{h,\tau}$ for diferent T; $h = \frac{1}{40}$, $\tau = \frac{1}{640}$.

		$T = 1$	$T = 3$	$T = 10$
$a = 1,\ \gamma = 1,\ b = 1$	$E^n_{h,\tau}$	$7.4133 \cdot 10^{-4}$	19.4167	56.2224
	$\varepsilon^n_{h,\tau}$	0.0071	1.2292	2.6604
$a = 1,\ \gamma = 2,\ b = 1$	$E^n_{h,\tau}$	$7.0765 \cdot 10^{-4}$	19.4165	56.1322
	$\varepsilon^n_{h,\tau}$	0.0150	2.7013	5.6986
$a = -1,\ \gamma = 1,\ b = 1$	$E^n_{h,\tau}$	$4.3511 \cdot 10^{-4}$	$4.3511 \cdot 10^{-4}$	$4.3511 \cdot 10^{-4}$
	$\varepsilon^n_{h,\tau}$	0.0209	0.0687	0.2320
$a = 0.2,\ \gamma = 1,\ b = 1$	$E^n_{h,\tau}$	$5.1908 \cdot 10^{-4}$	$6.3246 \cdot 10^{-4}$	0.0019
	$\varepsilon^n_{h,\tau}$	0.0079	0.0079	0.0079

8 Conclusions

The results obtained in the paper extend the class of the differential problems with nonlocal conditions when it is possible to prove the stability and convergence of difference schemes in the uniform norm. In this case, there is no need to require the matrix of the difference problem to be symmetrical or diagonally dominant. In this paper, the conditions of the stability and convergence are received using the main presumption that all the eigenvalues of the corresponding difference problem with nonlocal conditions are positive. Thus, nonlocal condition (2.3) can be interpreted as one particular case of nonlocal conditions when it is possible to prove the stability and convergence of the difference scheme in the uniform norm. So, the methodology of M-matrices might be also used in different cases of nonlocal conditions when all eigenvalues of the difference problem are positive. Some of such cases are examined in review article [37].

Furthermore, from the properties og M-matrices it follows that it would be enough to require even more general conditions. Namely, that the condition $\mathrm{Re}\lambda > 0$ should be true for all the eigenvalues of the corresponding matrix. In papers [30, 32, 34], several specified examples with other types of nonlocal conditions are provided, when there exist complex eigenvalues, the real parts of which are only positive. It might be possible to apply the methodology of the present paper to these problems. The relevant majorant $w(x, y, t)$ should be constructed.

Regarding, that estimates (5.17), and (6.7), (6.8), obtained in the present paper, depend on the constant ρ (see the hypothesis H2 in Section 2). It is not clear whether this constant should be certainly included in the estimate. It is only possible to state that the estimates depend on the choice of the majorant $w(x, y, t)$. In addition, the function $w_2(x, y)$ is very close to the majorant used in the maximum principle to the stationary problems [25].

References

[1] A. Berman and R.J. Plemmons. *Nonnegative Matrices in the Mathematical Sciences*. SIAM, Philadelphia, 1994. https://doi.org/10.1137/1.9781611971262.

[2] B. Cahlon, D.M. Kulkarni and P. Shi. Stepwise stability for the heat equation with a nonlocal constrain. *SIAM J. Numer. Anal.*, **32**(2):571–593, 1995. https://doi.org/10.1137/0732025.

[3] J.R. Cannon. The solution of the heat equation subject to specification of energy. *Q. Appl. Math.*, **21**(2):155–160, 1963. https://doi.org/10.1090/qam/160437.

[4] J.R. Cannon, Y. Lin and A.L. Matheson. The solution of the diffusion equation in two-space variables subject to specification of mass. *Appl. Anal.*, **50**(1):1–15, 1993. https://doi.org/10.1080/00036819308840181.

[5] Y.S. Choi and K.Y. Chan. A parabolic equation with nonlocal boundary conditions arising from electrochemistry. *Nonlinear Analysis, Theory, Method., Apli.*, **18**(4):317–331, 1992. https://doi.org/10.1016/0362-546X(92)90148-8.

[6] R. Čiegis, O. Suboč and A.Bugajev. Parallel algorithms for three-dimensional parabolic and pseudoparabolic problems with different boundary conditions. *Nonlinear. Anal. Model. Control*, **19**(3):382–395, 2014. https://doi.org/10.15388/NA.2014.3.5.

[7] R. Čiegis, O. Suboč and Rem. Čiegis. Numerical simulation of nonlocal delayed contraller for simple bioreactors. *Informatica*, **29**(2):233–249, 2018. https://doi.org/10.15388/Informatica.2018.165.

[8] R. Čiegis and N. Tumanova. Numerical solution of parabolic problems with nonlocal boundary conditions. *Numer. Funct. Anal. Optim.*, **31**(12):1318–1329, 2010. https://doi.org/10.1080/01630563.2010.526734.

[9] R. Čiegis, A. Štikonas, O. Štikonienė and O. Suboč. A monotonic finite-difference scheme for a parabolic problem with nonlocal condition. *Differ. Equ.*, **38**(7):1027–1037, 2002. https://doi.org/10.1023/A:1021167932414.

[10] W.A. Day. Extensions of a property of the heat equation to linear thermo-elasticity and order theories. *Q. Appl. Math.*, **40**:319–330, 1982. https://doi.org/10.1090/qam/678203.

[11] G. Ekolin. Finite-difference methods for a nonlocal boundary-value problem for heat equation. *BIT*, **31**:245–261, 1991. https://doi.org/10.1007/BF01931285.

[12] G. Fairweather and J.C. Lopez-Marcos. Galerkin methods for a semilinear parabolic problem with nonlocal conditions. *Adv. Comp. Math.*, **6**:243–262, 1996. https://doi.org/10.1007/BF02127706.

[13] A.V. Gulin, N.I. Ionkin and V.A. Morozova. Stability of a nonlocal two-dimensional finite-difference problem. *Differ. Equ.*, **37**(7):970–978, 2001. https://doi.org/10.1023/A:1011961822115.

[14] F. Ivanauskas, V. Laurinavičius, M. Sapagovas and A. Nečiporenko. Reaction-diffusion equation with nonlocal boundary condition subject to PID-controller bioreactor. *Nonlinear. Anal. Model. Control*, **22**(2):261–272, 2017. https://doi.org/10.15388/NA.2017.2.8.

[15] F. Ivanauskas, T. Meškauskas and M. Sapagovas. Stability of difference schemes for two-dimensional parabolic equations with nonlocal conditions. *Appl. Math. Comp.*, **215**(7):2716–2732, 2009. https://doi.org/10.1016/j.amc.2009.09.012.

[16] J. Jachimavičienė, Ž. Jesevičiūtė and M. Sapagovas. The stability of finite-difference schemes for a pseudoparabolic equation with nonlocal conditions. *Numer. Funct. Anal. Optim.*, **30**(9):988–1001, 2009. https://doi.org/10.1080/01630560903405412.

[17] J. Jachimavičienė, M. Sapagovas, A. Štikonas and O. Štikonienė. On the stability of explicit finite difference schemes for a pseudoparabolic equation with nonlocal conditions. *Nonlinear. Anal. Model. Control*, **19**(2):225–240, 2014. https://doi.org/10.15388/NA.2014.2.6.

[18] K. Jakubėlienė, R. Čiupaila and M. Sapagovas. Semi-implicit difference scheme for a two-dimensional parabolic equation with an integral boundary condition. *Math. Model. Anal.*, **22**(5):617–633, 2017. https://doi.org/10.3846/13926292.2017.1342709.

[19] L.I. Kamynin. A boundary value problem in the theory of the heat conduction with nonclassical boundary condition. *Zh. Vychisl. Mat. Mat. Fiz.*, **4**(6):1006–1024 (in Russian), 1964. https://doi.org/10.1016/0041-5553(64)90080-1.

[20] T. Leonavičienė, A. Bugajev, G. Jankevičiūtė and R. Čiegis. On stability analysis of finite difference schemes for generalized Kuramoto-Tsuzuki equation with nonlocal boundary conditions. *Math. Model. Anal.*, **21**(5):630–643, 2016. https://doi.org/10.3846/13926292.2016.1198836.

[21] Y. Lin, S. Xu and H. M. Yin. Finite difference approximations for a class of non-local parabolic equations. *Intern. J. Math. and Mat. Sci.*, **20**(1):147–164, 1997. https://doi.org/10.1155/S0161171297000215.

[22] J. Martin-Vaquero, A. Queiruga-Dios and A. H. Encinas. Numerical algorithms for diffusion-reaction problems with non-classical conditions. *Appl. Math. Compt.*, **218**(9):5487–5492, 2012. https://doi.org/10.1016/j.amc.2011.11.037.

[23] A.M. Nakhushev. On certain approximate method for boundary-value problems for differential equations and its applications in ground waters dynamics. *Differ. Uravn.*, **18**(1):72–81 (in Russian), 1988.

[24] C.V. Pao. Numerical solutions of reaction-diffusion equations with nonlocal boundary conditions. *J. Comp. Appi. Math.*, **136**:227–243, 2001. https://doi.org/10.1016/S0377-0427(00)00614-2.

[25] A.A. Samarskii. *The Theory of Difference Schemes.* Moscow, Nauka, 1977. https://doi.org/10.1201/9780203908518. (in Russian); Marcel Dekler, Inc., New York and Basel, 2001.

[26] M. Sapagovas. On the stability of a finite-difference scheme for nonlocal parabolic boundary-value problems. *Lith. Math. J.*, **48**(3):339–356, 2008. https://doi.org/10.1007/s10986-008-9017-5.

[27] M. Sapagovas. On the spectral properties of three-layer difference schemes for parabolic equations with nonlocal conditions. *Differ. Equ.*, **48**(7):1018–1027, 2012. https://doi.org/10.1134/S0012266112070129.

[28] M. Sapagovas, V. Griškonienė and O. Štikonienė. Application of M-matrices for solution of a nonlinear elliptic equation with an integral condition. *Nonlinear. Anal. Model. Control*, **22**(4):489–504, 2017. https://doi.org/10.15388/NA.2017.4.5.

[29] M. Sapagovas and K. Jakubėlienė. Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition. *Nonlinear. Anal. Model. Control*, **17**(1):91–98, 2012. https://doi.org/10.15388/NA.17.1.14080.

[30] M. Sapagovas, T. Meškauskas and F. Ivanauskas. Numerical spectral analysis of a difference operator with non-local boundary conditions. *Appl. Math. Comput.*, **218**(14):7515–7527, 2012. https://doi.org/10.1016/j.amc.2012.01.017.

[31] M. Sapagovas, T. Meškauskas and F. Ivanauskas. Influence of complex coefficients on the stability of difference scheme for parabolic equations with nonlocal conditions. *Appl. Math. Comp.*, **332**:228–240, 2018. https://doi.org/10.1016/j.amc.2018.03.072.

[32] M. Sapagovas, A. Štikonas and O. Štikonienė. Alternating direction method for the Poisson equation with weight coefficients in an integral condition. *Differ. Equ.*, **47**(8):1163–1174, 2011. https://doi.org/10.1134/S0012266111080118.

[33] M. Sapagovas, O. Štikonienė, K. Jakubėlienė and R. Čiupaila. Finite difference method for boundary value problem for nonlinear elliptic equation with nonlocal conditions. *Bound. Value Probl.*, **2019**(94):1–16, 2019. https://doi.org/10.1186/s13661-019-1202-4.

[34] M. Sapagovas, R. Čiupaila, Ž. Jokšienė and T. Meškauskas. Computational experiment for stability analysis of difference schemes with nonlocal conditions. *Informatica*, **24**(2):275–290, 2013.

[35] M.P. Sapagovas and A.D. Štikonas. On the stucture of the spectrum of a differential operator with a nonlocal condition. *Differ. Equ.*, **41**(7):1010–1018, 2005. https://doi.org/10.1007/s10625-005-0242-y.

[36] A. Štikonas. Investigation of characteristic curve for Sturm-Liouville problem with nonlocal boundary conditions on torus. *Math. Model. Anal.*, **16**(1):1–22, 2011. https://doi.org/10.3846/13926292.2011.552260.

[37] A. Štikonas. A survey on stationary problems, Green's functions and spectrum of Sturm-Liouville problem with nonlocal boundary conditions. *Nonlinear. Anal. Model. Control*, **19**(3):301–334, 2014. https://doi.org/10.15388/NA.2014.3.1.

[38] O. Štikonienė, M. Sapagovas and R. Čiupaila. On iterative methods for some elliptic equations with nonlocal conditions. *Nonlinear. Anal. Model. Control*, **19**(3):517–535, 2014. https://doi.org/10.15388/NA.2014.3.14.